Chapter 6

Differentiation

6.1 Differentiation of real functions of one real variable

6.1.1 *Derivative of a real function*

Definition 6.1.1 Let

- $f : X \subseteq \mathbb{R} \to \mathbb{R}$
- $x_0 \in X$
- x_0 be an accumulation point for X
- g be the function, called *incremental quotient at* x_0

$$(6.1.1) \qquad x \in X - \{x_0\} \quad \to \quad \frac{f(x) - f(x_0)}{x - x_0} .$$

Evidently, x_0 is an accumulation point for $X - \{x_0\}$ too. If there exists the limit at x_0 of the incremental quotient g and if such limit is a real number, we call such limit *derivative of the function* f *at the point* x_0, and denote it by $f'(x_0)$. So,

$$(6.1.2) \qquad f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R} .$$

We associate with the function f a real function f' whose domain is the set $X' \subseteq X$ of points $x_0 \in X$ at which the limit (6.1.2) exists. Such f' is called the *derivative* (or *first derivative*) of f. So,

$$f' \ : \ X' \subseteq X \to \mathbb{R} \ . \ \diamond$$

Remark 6.1.1 If $x \in X'$, we say that f is *differentiable at* x. If f' is defined at every point of a set $X' \subseteq X$, we say that f is *differentiable on* X'.

We underline that the derivative (or *first derivative*) of f *at any* $x \in X'$ *can be denoted, without distinction, with any of the symbols*

$$f'(x)\,, \quad Df(x) \ , \quad \frac{df}{dx}(x) \ .$$

Analogously, the derivative (or *first derivative*) of f *can be denoted, without distinction, with any of the symbols*

$$f' \ , \qquad Df \ , \qquad \frac{d}{dx}f \ , \qquad \frac{df}{dx} \ . \ \diamond$$

Theorem 6.1.1 *Let*

- $f \ : \ X \subseteq \mathbb{R} \to \mathbb{R}$
- $x_0 \in X$
- x_0 *be an accumulation point for* X .

The following statements are equivalent:

(6.1.3) $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \in \mathbb{R}$$

(6.1.4) $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \in \mathbb{R} \ .$$

Proof. Obvious. $\diamond$

Remark 6.1.2 The derivative (6.1.2) has a *geometric interpretation,* very important for the applications.
Let f be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. Let O, x, y be, in a plane, a system of orthogonal *Cartesian* axes. Consider, for each $x \in X$, the point P of the plane having abscissa x and ordinate $y = f(x)$. As x varies in X, the point $P = (x, f(x))$ describes in the plane a set of points called *Cartesian diagram of the function* f.

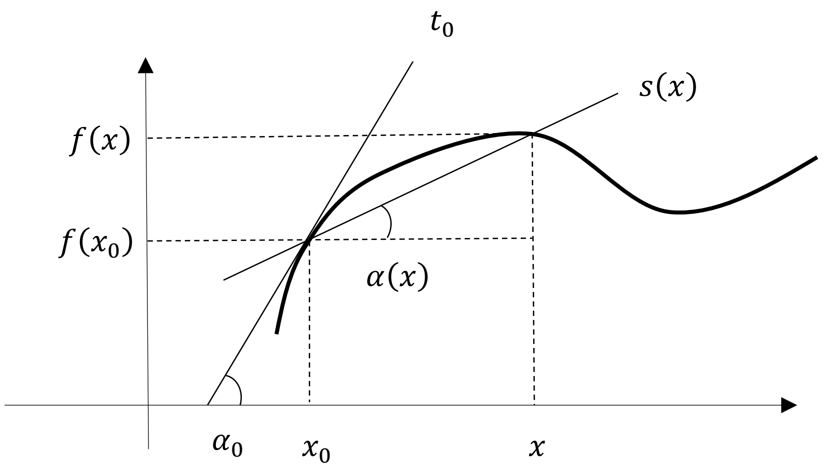

Fig. 6.1.1

Let (fig. 6.1.1)

- x_0 be a point of X at which f is differentiable
- x any point of X
- $s(x)$ the straight line passing through $(x_0, f(x_0))$ and *secant* the *Cartesian diagram* of the function f, at

point $(x, f(x))$

- t_0 a straight line *tangent* to *Cartesian diagram* of the function f in the point $(x_0, f(x_0))$.

Clearly the *angular coefficient* $\operatorname{tg}\alpha(x)$ of the secant straight line is just the incremental quotient at x_0

$$\operatorname{tg}\alpha(x) = \frac{f(x) - f(x_0)}{x - x_0} \ .$$

Let f be *differentiable at* x_0 . Then, we see in fig. 6.1.1 that when x tends to x_0 , the secant $s(x)$ tends to tangent t_0 . So,

$$\operatorname{tg}\alpha_0 = \lim_{x \to x_0} \operatorname{tg}\alpha(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0) \, .$$

Therefore, *if you say that* f *is differentiable in* x_0 *and the value of the derivative is* $f'(x_0)$ *, geometrically you say that*

- *the Cartesian diagram of the function* f *at the point* $(x_0, f(x_0))$ *has a tangent* t_0 *whose angular coefficient is just* $tg\ \alpha_0 = f'(x_0)$

- when x tends to x_0 , the secant $s(x)$ passing through $(x_0, f(x_0))$ and $(x, f(x))$ tends to tangent t_0 . ◊

Remark 6.1.3 The derivative of any *constant function* is clearly zero. ◊

Theorem 6.1.2 *Let*

- $f \ : \ X \subseteq \mathbb{R} \to \mathbb{R}$
- f *be differentiable at* $x_0 \in X$.

Then f *is continuous at* x_0 .

Proof. We have

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \cdot (x - x_0) \, .$$

Hence, by theorem 5.1.30

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = f'(x_0) \cdot 0 = 0 \, ,$$

hence, by theorem 5.1.29

$$\lim_{x \to x_0} f(x) = f(x_0) \ . \ \diamond$$

Remark 6.1.4 The converse of theorem 6.1.2 is not true. In fact, the *absolute value function*

$$|\cdot| \ : \ x \in \mathbb{R} \ \to \ |x| \in [0, +\infty[$$

is continuous in 0 , but its incremental quotient at 0 has not limit as $x \to 0$. In fact

$$\frac{|x| - |0|}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x \in \,]0, +\infty[\\ -1 & \text{if } x \in \,]-\infty, 0[\end{cases}$$

and then

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = -1 \quad \neq \quad \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = 1 \ . \ \diamond$$

On the derivative of the function sum there is the following theorem.

Theorem 6.1.3 *Let*

- $f\ :\ X \subseteq \mathbb{R} \to \mathbb{R}$
- $g\ :\ X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$.

Then $f + g$ *is differentiable at* x_0 *and it results*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0)\,. \tag{6.1.5}$$

Proof. By theorem 5.1.29 we have

$$(f + g)'(x_0) = \lim_{x \to x_0} \frac{(f + g)(x) - (f + g)(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0} + \frac{g(x) - g(x_0)}{x - x_0} \right) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

$$+ \lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = f'(x_0) + g'(x_0)\ . \ \diamond$$

On the derivative of the function product there is the following theorem.

Theorem 6.1.4 *Let*

- $f\ :\ X \subseteq \mathbb{R} \to \mathbb{R}$

- $g : X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$.

Then $f \cdot g$ *is differentiable at* x_0 *and it results*

(6.1.6) $$(f \cdot g)'(x_0) = f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0) .$$

Proof. By theorems 5.1.29, 5.1.30 and by continuity, we have

$$(f \cdot g)'(x_0) = \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \frac{f(x)\, g(x) - f(x_0)\, g(x) + f(x_0)\, g(x) - f(x_0)\, g(x_0)}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\left(\frac{f(x) - f(x_0)}{x - x_0} \right) g(x) + f(x_0) \left(\frac{g(x) - g(x_0)}{x - x_0} \right) \right]$$

$$= f'(x_0)\, g(x_0) + f(x_0)\, g'(x_0) \, . \ \diamond$$

Remark 6.1.5 Let

- $f : X \subseteq \mathbb{R} \to \mathbb{R}$
- $g : X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$
- $g(x_0) \neq 0$.

In such hypotheses, obviously there exists a neighborhood of x_0 where $\frac{f}{g}$ is defined and x_0 is accumulation point for the definition set of $\frac{f}{g}$. ⋄

On the derivative of the function quotient there is the following *theorem.*

Theorem 6.1.5 *Let*

- $f : X \subseteq \mathbb{R} \to \mathbb{R}$
- $g : X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$
- $g(x_0) \neq 0$.

Then $\frac{f}{g}$ *is differentiable at* x_0 *and it results*

$$(6.1.7) \qquad \left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)} \quad .$$

Proof. By theorems 6.1.2, 5.1.29, 5.1.30, 5.1.33 and by hypotheses, we have

$$\left(\frac{f}{g}\right)'(x_0) = \lim_{x \to x_0} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x \to x_0} \left[\frac{1}{g(x)g(x_0)}\left(\frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{x - x_0}\right)\right]$$

$$= \lim_{x \to x_0} \left\{\frac{1}{g(x)g(x_0)}\left[\left(\frac{f(x) - f(x_0)}{x - x_0}\right) g(x_0) - f(x)\left(\frac{g(x) - g(x_0)}{x - x_0}\right)\right]\right\}$$

$$= \frac{1}{g^2(x_0)}\left(f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)\right)$$

$$= \frac{f'(x_0)\, g(x_0) - f(x_0)\, g'(x_0)}{g^2(x_0)} \,. \quad \diamond$$

Remark 6.1.6 Let f be the *identical function*

$$f \ : \ x \in \mathbb{R} \to f(x) = x \, .$$

Then $\forall\, x \in \mathbb{R}$ it results $f'(x) = 1$. Moreover, repeated application of (6.1.6) shows that the function

$$f \ : \ x \in \mathbb{R} \to f(x) = x^n \, ,$$

where $n \in \mathbb{N}$, is differentiable on $\mathbb{R}$ and $\forall\, x \in \mathbb{R}$ it results

$$\frac{d}{dx}(x^n) = n\, x^{n-1} \ . \ \diamond$$

On the derivative of the composite function there is the following theorem.

Theorem 6.1.6 *Let*

- $f \ : \ X \subseteq \mathbb{R} \to \mathbb{R}$
- $g \ : \ f(X) \to \mathbb{R}$
- $x_0 \in X$
- x_0 *be accumulation point for* X
- $f(x_0)$ *be accumulation point for* $f(X)$
- f *be differentiable at* x_0
- g *be differentiable at* $f(x_0)$.

Then the composite function $g \circ f$ *is differentiable at* x_0 *and it results*

(6.1.8) $$(g \circ f)'(x_0) = g'\big(f(x_0)\big)\, f'(x_0) \, .$$

Proof. To obtain the (6.1.8), we have to prove that

(6.1.9) $\forall\, \varepsilon > 0 \quad \exists\, H_{x_0} \;:\; \forall\, x \in X \cap H_{x_0} - \{x_0\}$

$$\left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} - g'\big(f(x_0)\big)\, f'(x_0)\right| < \varepsilon\,.$$

To this aim, let us notice that

- by hypothesis, g is differentiable in $f(x_0)$, and then

(6.1.10) $\forall\, \varepsilon > 0 \quad \exists\, \delta_1 > 0 \;:\; \forall\, y \in f(X)$

$$(0 < |y - f(x_0)| < \delta_1) \Rightarrow \left(\left|\frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} - g'\big(f(x_0)\big)\right| < \varepsilon\right).$$

- by hypothesis, f is differentiable in x_0 , and then

(6.1.11) $\forall\, \rho > 0 \quad \exists\, W_{x_0} \;:\; \forall\, x \in X \cap W_{x_0} - \{x_0\}$

$$\left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right| < \rho\,.$$

- by theorem 6.1.2, f is continuous in x_0 , and then

(6.1.12) $\forall\, \beta > 0 \quad \exists\, J_{x_0} \;:\; \forall\, x \in X \cap J_{x_0} - \{x_0\}$

$$|f(x) - f(x_0)| < \beta\,.$$

Possible cases are $f'(x_0) \neq 0$ and $f'(x_0) = 0$.
Suppose $f'(x_0) \neq 0$. By theorem 5.1.11

$$\exists\, I_{x_0} \;:\; \forall\, x \in X \cap I_{x_0} - \{x_0\} \qquad \frac{f(x) - f(x_0)}{x - x_0} \neq 0\,,$$

hence

$$\exists\, I_{x_0} \quad : \quad \forall\, x \in X \cap I_{x_0} - \{x_0\} \qquad f(x) - f(x_0) \neq 0\,,$$

hence, $\forall\, x \in X \cap I_{x_0} - \{x_0\}$

(6.1.13) $$\left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} - g'\big(f(x_0)\big)\, f'(x_0)\right|$$

$$= \left|\left(\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)}\right)\left(\frac{f(x) - f(x_0)}{x - x_0}\right) - g'\big(f(x_0)\big)\, f'(x_0)\right|$$

$$= \left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)}\frac{f(x) - f(x_0)}{x - x_0} - \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)} f'(x_0)\right.$$

$$\left. + \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)} f'(x_0) - g'\big(f(x_0)\big)\, f'(x_0)\right|$$

$$\leq \left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)}\right| \cdot \left|\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0)\right|$$

$$+ |f'(x_0)| \cdot \left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)} - g'\big(f(x_0)\big)\right|.$$

Moreover, in correspondence of the positive real number 1 , by (6.1.10) we have

(6.1.14) $$\exists\, \delta_2 > 0 \quad : \quad \forall\, y \in f(X) \qquad (\, 0 < |y - f(x_0)| < \delta_2) \Rightarrow$$

$$\left|\frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)}\right| \leq \left|\frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} - g'\big(f(x_0)\big)\right|$$

$$+ \left|g'\big(f(x_0)\big)\right| < 1 + \left|g'\big(f(x_0)\big)\right| \quad .$$

To gain the (6.1.9), let us consider now any $\varepsilon \in\,]0\,, +\infty[$. We notice that, in correspondence of the positive real number

$$\frac{\varepsilon}{2\left(1 + \left|g'\big(f(x_0)\big)\right|\right)}\ ,$$

by (6.1.11) we have

(6.1.15) $\exists\, W_{x_0} \;:\; \forall\, x \in X \cap W_{x_0} - \{x_0\}$

$$\left|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)\right| < \frac{\varepsilon}{2\left(1+\left|g'\big(f(x_0)\big)\right|\right)} \quad .$$

Moreover, in correspondence of the positive real number

$$\frac{\varepsilon}{2\,|f'(x_0)|}\ ,$$

by (6.1.10) we have

(6.1.16) $\exists\, \delta_1 > 0 \;:\; \forall\, y \in f(X)$ it results

$$(0 < |y - f(x_0)| < \delta_1) \Rightarrow \left(\left|\frac{g(y)-g\big(f(x_0)\big)}{y-f(x_0)} - g'\big(f(x_0)\big)\right| < \frac{\varepsilon}{2\,|f'(x_0)|}\right).$$

Moreover, in correspondence of the positive real number $\delta = \min\{\delta_1\,, \delta_2\}$, by (6.1.12) we have

(6.1.17) $\exists\, J_{x_0} \;:\; \forall\, x \in X \cap J_{x_0} - \{x_0\} \quad |f(x) - f(x_0)| < \delta.$

Evidently, $H_{x_0} = I_{x_0} \cap W_{x_0} \cap J_{x_0}$ is a neighborhood of x_0 and for every $x \in X \cap H_{x_0} - \{x_0\}$, taking into account the (6.1.13), (6.1.17), (6.1.14), (6.1.15), (6.1.16), we have

$$\left|\frac{g\big(f(x)\big)-g\big(f(x_0)\big)}{x-x_0} - g'\big(f(x_0)\big)\,f'(x_0)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\ .$$

So, if $f'(x_0) \neq 0$ the (6.1.8) is true.

Suppose now $f'(x_0) = 0$. Obviously the (6.1.9) turns in

(6.1.18) $\forall\, \varepsilon > 0 \quad \exists\, H_{x_0} \;:\; \forall\, x \in X \cap H_{x_0} - \{x_0\}$

$$\left|\frac{g(f(x)) - g(f(x_0))}{x - x_0}\right| < \varepsilon .$$

To gain the (6.1.18), let us consider any $\varepsilon \in]0, +\infty[$. In correspondence of the positive real number 1, by (6.1.10) we have

(6.1.19) $\exists\, \delta > 0 \;:\; \forall\, y \in f(X) \quad (0 < |y - f(x_0)| < \delta) \Rightarrow$

$$\left|\frac{g(y) - g(f(x_0))}{y - f(x_0)}\right| < 1 + |g'(f(x_0))| .$$

Moreover, in correspondence of the positive real number δ, by (6.1.12) we have

(6.1.20) $\exists\, J_{x_0} \;:$

$$(x \in X \cap J_{x_0} - \{x_0\}) \;\Rightarrow\; (|f(x) - f(x_0)| < \delta) .$$

We notice that, in correspondence of the positive real number

$$\frac{\varepsilon}{1 + |g'(f(x_0))|} ,$$

by (6.1.11) we have

(6.1.21) $\exists\, W_{x_0} \;:$

$$(x \in X \cap W_{x_0} - \{x_0\}) \Rightarrow \left(\left|\frac{f(x) - f(x_0)}{x - x_0}\right| < \frac{\varepsilon}{1 + |g'(f(x_0))|}\right).$$

Evidently, $H_{x_0} = W_{x_0} \cap J_{x_0}$ is a neighborhood of x_0. Let $x \in X \cap H_{x_0} - \{x_0\}$. If $f(x) = f(x_0)$, obviously the (6.1.18) is true. If $f(x) \neq f(x_0)$, taking into account the (6.1.20), (6.1.19), (6.1.21), we have

$$\left|\frac{g(f(x)) - g(f(x_0))}{x - x_0}\right|$$
$$= \left|\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)}\right| \cdot \left|\frac{f(x) - f(x_0)}{x - x_0}\right|$$
$$< \left(1 + |g'(f(x_0))|\right)\frac{\varepsilon}{1 + |g'(f(x_0))|} = \varepsilon$$

and then the (6.1.18) is true. ⋄

On the derivative of the inverse function there is the following theorem.

Theorem 6.1.7 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a, b] \to \mathbb{R}$
- f *be continuous and strictly monotone*
- $x_0 \in [a, b]$
- f *be differentiable at* x_0
- $f'(x_0) \neq 0$.

Then the inverse function f^{-1} *is differentiable at* $f(x_0)$ *and it results*

(6.1.22) $$(f^{-1})'(f(x_0)) = \frac{1}{f'(x_0)} \ .$$

Proof. To gain the (6.1.22), we have to prove that

$$\lim_{y \to f(x_0)} \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} = \frac{1}{f'(x_0)}$$

i.e., that

(6.1.23) $\forall\, \varepsilon > 0 \quad \exists\, \delta > 0 \quad : \quad \forall\, y \in f(X)$

$$(0 < |y - f(x_0)| < \delta) \Rightarrow \left(\left| \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon \right).$$

Preliminarily we notice that, by hypothesis, $f(x_0)$ is an accumulation point for $f(X)$. In fact, suppose $I_{f(x_0)}$ be any neighborhood of $f(x_0)$. Because f is continuous in x_0, $\exists\, J_{x_0}$ such that $\forall\, x \in J_{x_0} \cap X - \{x_0\}$ it results $f(x_0) \in I_{f(x_0)}$. Since x_0 is an accumulation point for X, $\exists\, x_1 \in J_{x_0} \cap X - \{x_0\}$ and then $f(x_1) \in I_{f(x_0)}$. Since f is strictly monotonic, we have $f(x_1) \neq f(x_0)$ and then $I_{f(x_0)} \cap f(X) - \{f(x_0)\} \neq \emptyset$. Let us notice now that the function incremental quotient of f at x_0 converges toward a nonzero limit. As a consequence

$$\lim_{x \to x_0} \frac{1}{\left(\dfrac{f(x) - f(x_0)}{x - x_0} \right)} = \frac{1}{f'(x_0)}$$

i.e.,

(6.1.24) $\forall\, \varepsilon > 0 \quad \exists\, \delta_1 > 0 \quad : \quad \forall\, x \in X$

$$(0 < |x - x_0| < \delta_1) \Rightarrow \left(\left| \frac{1}{\left(\dfrac{f(x) - f(x_0)}{x - x_0} \right)} - \frac{1}{f'(x_0)} \right| < \varepsilon \right).$$

Moreover f^{-1} is continuous, since it is the inverse function of a continuous function. Hence

$$\lim_{y \to f(x_0)} f^{-1}(y) = f^{-1}\big(f(x_0)\big) = x_0$$

i.e.,

(6.1.25) $\quad \forall\, \beta > 0 \quad \exists\, \delta > 0 \quad : \quad \forall\, y \in f(X)$

$$(|y - f(x_0)| < \delta) \Rightarrow (|f^{-1}(y) - x_0| < \beta)\,.$$

Now we can build the (6.1.23). Let ε be any positive real number. The (6.1.24) gives us a positive real number δ_1 such that $\forall\, x \in X$

(6.1.26) $\quad (0 < |x - x_0| < \delta_1)$

$$\Rightarrow \left(\left| \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} - \frac{1}{f'(x_0)} \right| < \varepsilon \right).$$

In correspondence of the positive real number δ_1 , the (6.1.25) gives us a positive real number δ such that

(6.1.27) $\quad \forall\, y \in f(X) \quad (|y - f(x_0)| < \delta)$

$$\Rightarrow (|f^{-1}(y) - x_0| < \delta_1)\,.$$

Let us consider now any $y \in f(X)$ such that $0 < |y - f(x_0)| < \delta$. Putting $x = f^{-1}(y)$, from the (6.1.27) we obtain $|x - x_0| < \delta_1$. Furthermore $x \neq x_0$. In fact, by absurd, let us suppose $x = x_0$. Hence $y = f(x) = f(x_0)$. Absurd, since $0 < |y - f(x_0)|$. So, $0 < |x - x_0| < \delta_1$ and then from (6.1.26) we have

$$\left|\frac{1}{\left(\frac{f(x)-f(x_0)}{x-x_0}\right)}-\frac{1}{f'(x_0)}\right|<\varepsilon$$

from which

$$\left|\frac{x-x_0}{f(x)-f(x_0)}-\frac{1}{f'(x_0)}\right|<\varepsilon$$

from which

$$\left|\frac{f^{-1}(y)-f^{-1}\big(f(x_0)\big)}{y-f(x_0)}-\frac{1}{f'(x_0)}\right|<\varepsilon\,. \ \diamond$$

We define now the derivatives and the differentials of higher order.

Definition 6.1.2 Let

- $f \ : \ X \subseteq \mathbb{R} \to \mathbb{R}$
- $f' \ : \ X' \subseteq X \to \mathbb{R}$.

If there exists the derivative $(f')'$ of f', we denote it f'' and call it the *second derivative* of f . So

(6.1.28) $$f'' = (f')' \ : \ X'' \subseteq X' \to \mathbb{R}\,. \ \diamond$$

Definition 6.1.3 Let $f \ : \ X \subseteq \mathbb{R} \to \mathbb{R}$. By induction, the nth *derivative* $f^{(n)}$ (or *derivative of order n*) of f is defined as the first derivative of the derivative $f^{(n-1)}$ of the $(n-1)$th order. So

(6.1.29) $$f^{(n)} = \left(f^{(n-1)}\right)' \ : \ X^{(n)} \subseteq X^{(n-1)} \to \mathbb{R}\,. \ \diamond$$

Remark 6.1.7 Of course, the nth derivative of a function f at a point or at subset of X may or may not exist. ⋄

Definition 6.1.4 Let $f\ :\ X\subseteq\mathbb{R}\to\mathbb{R}$ be a function differentiable at $x\in X$. We call *differential* (or *first differential*) *of* f at x, and denote df, the first degree polynomial

(6.1.30) $$df\ :\ dx\in\mathbb{R}\to df(dx)=f'(x)\cdot dx\ \in\mathbb{R}.$$ ⋄

Remark 6.1.8 In the (6.1.30), dx is often called the *differential of* x. ⋄

Definition 6.1.5 Let $f\ :\ X\subseteq\mathbb{R}\to\mathbb{R}$, $x\in X$. We call *increment* Δf *of* f at x the function

(6.1.31) $$\Delta f\ :\ dx\in\mathbb{R}\to\ \Delta f(dx)=f(x+dx)-\ f(x)\in\mathbb{R}.$$ ⋄

Remark 6.1.9 We observe that Δf and df are both infinitesimal in 0. ⋄

Theorem 6.1.8 *Let*

- $f\ :\ X\subseteq\mathbb{R}\to\mathbb{R}$
- f *be differentiable at* $x\in X$.

Then $\Delta f-df$ *is an infinitesimal of higher order than* 1 *, i.e.,* [6.1.1]

[6.1.1] See definition 5.1.46.

(6.1.32) $$\lim_{dx\to 0}\frac{\Delta f(dx)-df(dx)}{dx}=0\,.$$

Proof. In fact, it results

$$\lim_{dx\to 0}\frac{\Delta f(dx)-df(dx)}{dx}$$
$$=\lim_{dx\to 0}\left(\frac{f(x+dx)-f(x)}{dx}-f'(x)\right)=0\,.\ \diamond$$

Remark 6.1.10 Theorem 6.1.8 allows us to approximate, in a convenable neighborhood of $x\in X$, any function f differentiable at x. In fact (fig. 6.1.2)

- let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ any function differentiable at $x\in X$
- let I_x any neighborhood of x
- *linearize* in I_x the function f, i.e. approximate in I_x f by replacing f with the tangent t at point $(x, f(x))$ of the *Cartesian diagram* of f (fig. 6.1.2)
- notice (see fig. 6.1.2) that the point of t having abscissa $x+dx$, has ordinate $t(x+dx)=f(x)+tg\ \alpha\cdot dx = f(x)+f'(x)\cdot dx=f(x)+df(dx)$
- call *error* of the approximation the function $e : x+dx\in I_x\cap X\to e(x+dx)=f(x+dx)-t(x+dx)=f(x+dx)-f(x)-df(dx)=\Delta f(dx)-df(dx)$
- notice that in fig. 6.1.2 the segment P_2P_4 has length $t(x+dx)$, the segment P_1P_4 has length $f(x+dx)$, the segment P_1P_2 has length $e(x+dx)$, the segment

P_1P_3 has length $df(dx) = f'(x) \cdot dx$, the segment P_5P_4 (or P_6P_3) has length dx, the segment P_3P_4 (or P_6P_5) has length $f(x)$

- notice that the error e is infinitesimal in 0 and, by theorem 6.1.8, tends to zero faster than the size of the neighborhood I_x.

So, in a convenable neighborhood of x, the linearization error is negligible. ⬨

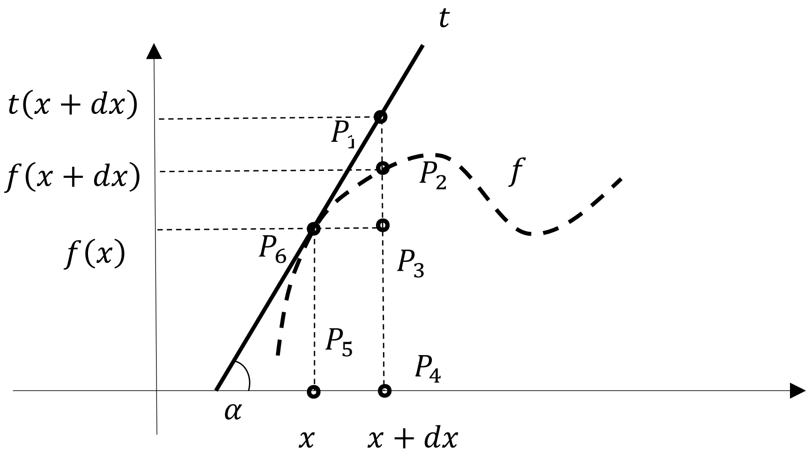

Fig. 6.1.2

Remark 6.1.11 We underline that, since the (6.1.30) establishes that

$$df = f'(x) \cdot dx\,,$$

it results

(6.1.33) $$f'(x) = \frac{df}{dx} \ .$$

So, *the derivative of the function* f *with respect to the variable* x *is equal to ratio of the corresponding differentials.* ◊

Definition 6.1.6 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable at $x \in X$. By induction, we call n-th *differential* (or *differential of the* n-th *order*) *of* f *at* x , and denote $d^n f$, the n-th degree polynomial

(6.1.34) $$d^n f = d(d^{n-1} f) = d\big(f^{(n-1)}(x) \cdot dx^{n-1}\big)$$

$$= f^{(n)}(x) \cdot dx^n \ : \ dx \in \mathbb{R} \to d^n f(dx) = f^{(n)}(x) \cdot dx^n \in \mathbb{R}$$

where dx^n means $(dx)^n$. ◊

Remark 6.1.12 It follows from equality (6.1.34) that the n-th *derivative of the function* f *is equal to ratio of* $d^n f$ *to* $(dx)^n$, *i.e.*,

(6.1.35) $$f^{(n)} = \frac{d^n f}{dx^n} \ . \ \diamond$$

6.1.2 *Mean value theorems*

Definition 6.1.7 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f has a *local maximum* at point $x_0 \in X$ if

(6.1.36) $$\exists \ I_{x_0} \ : \ \forall \, x \in I_{x_0} \cap X \qquad f(x) \le f(x_0)$$

where I_{x_0} denotes a neighborhood of x_0 . ◊

Definition 6.1.8 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f has a *proper local maximum* at point $x_0 \in X$ if

(6.1.37) $$\exists\ I_{x_0} : \forall\, x \in I_{x_0} \cap X - \{x_0\} \qquad f(x) < f(x_0)$$

where I_{x_0} denotes a neighborhood of x_0 . ◊

Definition 6.1.9 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f has a *local minimum* at point $x_0 \in X$ if

(6.1.38) $$\exists\, I_{x_0} : \forall\, x \in I_{x_0} \cap X \qquad f(x) \geq f(x_0)$$

where I_{x_0} denotes a neighborhood of x_0 . ◊

Definition 6.1.10 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. We say that f has a *proper local minimum* at point $x_0 \in X$ if

(6.1.39) $$\exists\, I_{x_0} : \forall\, x \in I_{x_0} \cap X - \{x_0\} \qquad f(x) > f(x_0)$$

where I_{x_0} denotes a neighborhood of x_0 . ◊

Definition 6.1.11 Let $f : X \subseteq \mathbb{R} \to \mathbb{R}$. Local maxima and minima are also called *local extrema*. ◊

Theorem 6.1.9 [*Fermat* [6.1.2]] *Suppose*

- $f : X \subseteq \mathbb{R} \to \mathbb{R}$
- f *have a local extremum at* $x_0 \in X$

[6.1.2] *Pierre Fermat*, Beaumont-de-Lomagne (France) 1601 – Castres 1665.

- f *be differentiable at* x_0 .

Then, it results

- *if* x_0 *is accumulation point for* X *both on the right and on the left*

$$f'(x_0) = 0 \tag{6.1.40}$$

- *if* x_0 *is accumulation point for* X *but is not accumulation point on the right, and if* x_0 *is a point of local maximum*

$$f'(x_0) \geq 0 \tag{6.1.41}$$

- *if* x_0 *is accumulation point for* X *but is not accumulation point on the right, and if* x_0 *is a point of local minimum*

$$f'(x_0) \leq 0 \tag{6.1.42}$$

- *if* x_0 *is accumulation point for* X *but is not accumulation point on the left, and if* x_0 *is a point of local maximum*

$$f'(x_0) \leq 0 \tag{6.1.43}$$

- *if* x_0 *is accumulation point for* X *but is not accumulation point on the left, and if* x_0 *is a point of local minimum*

(6.1.44) $$f'(x_0) \geq 0 .$$

Proof. Let

- ✓ x_0 be accumulation point for X both on the right and on the left
- ✓ x_0 be point of local maximum for f .

To prove the (6.1.40), reasoning by absurd, suppose $f'(x_0) > 0$. Hence

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0 ,$$

hence, by theorem 5.1.11

(6.1.45) $$\exists\, I_{x_0} \ : \quad \left(x \in I_{x_0} \cap X - \{x_0\} \right) \Rightarrow \left(\frac{f(x) - f(x_0)}{x - x_0} > 0 \right) .$$

Moreover, by hypothesis

(6.1.46) $$\exists\, J_{x_0} \ : \ \forall\, x \in J_{x_0} \cap X - \{x_0\} \qquad f(x) \leq f(x_0) .$$

Since definition 3.1.7 and theorem 3.1.8, there exists $r \in]0 , +\infty[$ such that $N_{x_0} =]x_0 - r ,\ x_0 + r[\subseteq I_{x_0} \cap J_{x_0}$. So, N_{x_0} is a neighborhood of x_0 and, taking into account the (6.1.45) and (6.1.46), we have

(6.1.47) $$\forall\, x \in N_{x_0} \cap X - \{x_0\} \qquad \begin{cases} \dfrac{f(x) - f(x_0)}{x - x_0} > 0 \\ f(x) \leq f(x_0) . \end{cases}$$

By hypothesis, x_0 is accumulation point on the right for X

and then there exists $z \in [x_0 , x_0 + r[\cap X - \{x_0\} \subseteq N_{x_0} \cap X - \{x_0\}$. As a consequence $z > x_0$ and, by (6.1.47), we simultaneously have $f(z) > f(x_0)$ and $f(z) \leq f(x_0)$. Absurd. Then the inequality $f'(x_0) > 0$ is impossible.
With a similar reasoning we prove that also the inequality $f'(x_0) < 0$ is impossible. So, we have proven that the (6.1.40) is true.
If the local extremum of f is a local minimum, with a reasoning very similar to the previous one we still prove that the (6.1.40) is true.
After that, it is easy to obtain, similarly still reasoning by absurd, the other statements, i.e. $(6.1.41), (6.1.42), (6.1.43), (6.1.44)$. ⋄

Real functions differentiable in intervals have the following notable properties.

Theorem 6.1.10 [*Rolle*] *Let*

- $a , b \in \mathbb{R}$
- $a < b$
- $f \,:\, [a , b] \rightarrow \mathbb{R}$
- f *be continuous in* $[a , b]$
- f *be differentiable in* $]a , b[$
- $f(a) = f(b)$.

Then, there exists $c \in \,]a, b[$ *such that*

$$f'(c) = 0 .$$

Proof. Since $[a , b]$ is a closed and bounded subset of $\mathbb{R}$, by theorem 5.3.6 there exist $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$. Obviously

(6.1.48) $\forall\, x \in [a\,,b] \qquad m \leq f(x) \leq M\,.$

If $m = M$, the (6.1.48) implies that f is constant and then for every $c \in \,]a\,,b[$ we have $f'(c) = 0\,.$
If $m \neq M$ obviously it results $m < M$. Let $h\,,k$ be points of $[a\,,b]$ such that $m = f(h)\,,\ M = f(k)$. By hypothesis $f(a) = f(b)$ and then at least one of the points $a\,,b$ belongs to $]a\,,b[$. We call c this point and notice that

- c is point of absolute extremum for f , and then is point of local extremum for f
- $c \in \,]a\,,b[$ and then f is differentiable at c
- $c \in \,]a\,,b[$ and then is accumulation point for $[a\,,b]$ both on the right and on the left.

So, by theorem 6.1.9, we have $f'(c) = 0\,.$ ◊

Theorem 6.1.11 [*Lagrange* [6.1.3]] *Let*

- $a\,,b \in \mathbb{R}$
- $a < b$
- $f\ :\ [a\,,b] \to \mathbb{R}$
- f *be continuous in* $[a\,,b]$
- f *be differentiable in* $]a\,,b[\,.$

[6.1.3] *Joseph-Louis Lagrange*, Torino (Italia) 25.01.1736 – Paris 10.04.1813

Then, there exists $c \in]a\,,b[$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad . \tag{6.1.49}$$

Proof. Let us consider the function g that to every $x \in [a\,,b]$ associates the real number

$$f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a)\,.$$

Evidently g is continuous in $[a\,,b]$ and differentiable in $]a\,,b[$. Moreover, it results $g(b) = g(a) = 0$. So, by theorem 6.1.10, there exists $c \in]a\,,b[$ such that $g'(c) = 0$. Hence

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0$$

and then the (6.1.49) is true. ◊

Theorem 6.1.12 [*Cauchy*] *Let*

- $a\,,b \in \mathbb{R}$
- $a < b$
- $f\ :\ [a\,,b] \to \mathbb{R}$
- $g\ :\ [a\,,b] \to \mathbb{R}$
- f *and* g *be continuous in* $[a\,,b]$
- f *and* g *be differentiable in* $]a, b[$.

Then there exists $c \in]a\,,b[$ *such that*

(6.1.50) $\qquad f'(c)\big(g(b)-g(a)\big)=g'(c)\big(f(b)-f(a)\big)\,.$

Proof. Let us consider the function h that to every $x\in[a\,,b]$ associates the real number

$$f(x)\big(g(b)-g(a)\big)-g(x)\big(f(b)-f(a)\big)\,.$$

Evidently h is continuous in $[a\,,b]$ and differentiable in $]a\,,b[$. Moreover, it results $h(a)=h(b)$. So, by theorem 6.1.10, there exists $c\in\,]a\,,b[$ such that $h'(c)=0$. Hence

$$f'(c)\big(g(b)-g(a)\big)-g'(c)\big(f(b)-f(a)\big)=0$$

and then the (6.1.50) is true. ◊

Theorem 6.1.13 *Let*

- $a\,,b\in\mathbb{R}$
- $a<b$
- $f\ :\ [a\,,b]\to\mathbb{R}$
- f *be continuous in* $[a\,,b]$
- f *be differentiable in* $]a\,,b[$
- $\forall\,x\in\,]a\,,b[\qquad f'(x)=0\,.$

Then

(6.1.51) $\qquad \exists\,k\in\mathbb{R}\ \ :\ \ f(x)=k\qquad\forall\,x\in[a\,,b]\,.$

Proof. Let us consider any $x\in\,]a\,,b[$. If $x=a$, it results $f(x)=f(a)$. If $x\neq a$, it results $a<x$. So, the *restriction* of f to $[a\,,x]$ satisfies the hypotheses of theorem 6.1.11.

Consequently, there exists $c \in \,]a\,,x[$ such that

$$f'(c) = \frac{f(x) - f(a)}{x - a} \quad ,$$

hence, since by hypothesis $f'(c) = 0$, we obtain $f(x) = f(a)$. In this way, the $(6.1.51)$ is proven. ◊

Remark 6.1.13 Theorem 6.1.13 is valid only for real functions defined in intervals. Otherwise, the function defined in a generic numerical set X results only constant at times, i.e. constant only in every interval contained in X. ◊

Theorem 6.1.14 *Let*

- $a\,,b \in \mathbb{R}$
- $a < b$
- $f \;:\; [a\,,b] \to \mathbb{R}$
- $g \;:\; [a\,,b] \to \mathbb{R}$
- f *and* g *be continuous in* $[a\,,b]$
- f *and* g *be differentiable in* $]a\,,b[$
- $\forall\, x \in \,]a\,,b[\qquad f'(x) = g'(x)$.

Then, there exists $c \in \mathbb{R}$ *such that*

$$(6.1.52) \qquad \forall\, x \in [a\,,b] \qquad f(x) = g(x) + c\,.$$

Proof. We observe that the function $f - g$ satisfies all hypotheses of theorem 6.1.13. So, there exists $c \in \mathbb{R}$ such that the $(6.1.52)$ is true. ◊

Theorem 6.1.15 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a, b] \to \mathbb{R}$
- f *be continuous in* $[a, b]$
- f *be differentiable in* $]a, b[$
- $\forall\, x \in\,]a, b[\qquad f'(x) > 0$.

Then f *is strictly increasing in* $[a, b]$.

Proof. Let us consider any $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$. Clearly, the restriction of f to $[x_1, x_2]$ satisfies the hypotheses of theorem 6.1.11. Consequently, there exists $c \in\,]x_1, x_2[$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) > 0 .$$

Hence $f(x_1) < f(x_2)$. ◊

Theorem 6.1.16 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a, b] \to \mathbb{R}$
- f *be continuous in* $[a, b]$
- f *be differentiable in* $]a, b[$
- $\forall\, x \in\,]a, b[\qquad f'(x) < 0$.

Then f *is strictly decreasing in* $[a, b]$.