CHAPTER 10

ORDINARY DIFFERENTIAL EQUATIONS ◊

10.1 First order differential equations

10.1.1 Introduction

Definition 10.1.1 Let $k, n \in \mathbb{N}$, $\Omega \subseteq \mathbb{R}^k$, g be a known real function defined on Ω. We call *differential equation of order* n the problem of finding a function y such that

- $y : \Omega \to \mathbb{R}$
- y is provided on Ω of all derivatives up those of order n included
- y satisfies an equation where there are y, at least one of its derivatives of order n and the known term g.

If $k > 1$ the differential equation of order n is called *partial differential equation of order* n .
If $k = 1$ the differential equation of order n is called *ordinary differential equation of order* n . ◊

Remark 10.1.1 Important examples of partial differential equation are

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- the *Laplace's equation* $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} = 0$
- the *heat equation* $a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial w}{\partial t}$
- the *wave equation* $a^2 \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) = \frac{\partial^2 w}{\partial z^2}$. ◊

Definition 10.1.2 Let $n \in \mathbb{N}$. We say that an ordinary differential equation of order n it's in *normal form* if it's of the type

$$y^{(n)} = f\left(x, y, y', \dots, y^{(n-1)}\right)$$

where

$$x \in [a, b] \subseteq \mathbb{R}$$

$$y : [a, b] \to \mathbb{R}$$

$$y' = y^{(1)} = \frac{d}{dx} y$$

$$y^{(2)} = \frac{d^2}{dx^2} y$$

$$\dots$$

$$y^{(n)} = \frac{d^n}{dx^n} y \ . \ \diamond$$

Remark 10.1.2 In this chapter we only study the ordinary differential equation of order 1 . ◊

Definition 10.1.3 Every solution of a differential equation is also called an *integral*, or a *particular integral*, of the differential equation. ◊

Definition 10.1.4 If there exists a unique expression, where an arbitrary constant appears, that represents all and only the solutions of the differential equation, we call the expression the *general integral* of the differential equation. ◊

Definition 10.1.5 If there exists a unique expression, where an arbitrary constant appears, that represents almost all solutions of the differential equation, we call *singular integral* of the differential equation every particular integral which is not represented by the expression. ◊

10.1.2 Linear differential equations

Definition 10.1.6 A first order ordinary differential equation is called *separable* if it can be written in the form

$$y' = f(x)h(y) \quad . \qquad (10.1.1)$$ ◊

Remark 10.1.3 The separable first order ordinary differential equation $(10.1.1)$ can be written

$$\frac{dy}{dx} = f(x)h(y)$$

from which, by separating the variables, we obtain

$$\frac{1}{h(y)}\frac{dy}{dx} = f(x)\,. \qquad (10.1.2)$$

By integration, the $(10.1.2)$ can give the solutions of $(10.1.1)$. ◊

Example 10.1.1 The separable first order ordinary differential equation on the basic interval $[a, b]$

$$yy' = x \tag{10.1.2}$$

can be written

$$y\frac{dy}{dx} = x$$

and then

$$\frac{d}{dx}\left(\frac{y^2}{2}\right) = x\,.$$

As a consequence, in $[a, b]$ the function x admits the primitive $\frac{y^2}{2}$ and the primitive $\int_a^x t\ dt = \frac{x^2}{2} + c_1$, where $c_1 \in \mathbb{R}$. Hence there exists $c \in \mathbb{R}$ such that

$$\frac{y^2}{2} - \frac{x^2}{2} = c$$

hence $y = \sqrt{x^2 + 2c}$ and $y = -\sqrt{x^2 + 2c}$ are two solutions of the (10.1.2). ◊

Definition 10.1.7 We call *linear first order ordinary differential equation* any first order ordinary differential equation in which y and y' appear at first power. So, in the general case it is written

$$y' + a(x)y = f(x) \tag{10.1.3}$$

where x belongs to open interval Ω of $\mathbb{R}$ and the real functions a, f are defined on Ω.

The $(10.1.3)$ is also called the *complete differential equation.* ◊

Definition 10.1.8 If the known term is zero, the $(10.1.3)$ becomes

$$(10.1.4) \qquad y' + a(x)y = 0$$

and is called the *homogeneous differential equation associated* to complete differential equation $(10.1.3)$. ◊

Theorem 10.1.1 *Let*

- Ω *be an open interval of* $\mathbb{R}$
- $a : \Omega \to \mathbb{R}$
- $f : \Omega \to \mathbb{R}$
- *a, f be continuous*
- $x_0 \in \Omega$.

Then

$$(10.1.5) \qquad y = ce^{-\int_{x_0}^{x} a(t)\, dt} + e^{-\int_{x_0}^{x} a(t)\, dt} \int_{x_0}^{x} \left[f(t)\, e^{\int_{x_0}^{t} a(z)\, dz} \right] dt$$

is the general integral of the complete differential equation $(10.1.3)$.

Proof. To prove that $(10.1.5)$ is the general integral of the $(10.1.3)$, we have to prove that

1) for every $c \in \mathbb{R}$, the $(10.1.5)$ is solution of $(10.1.3)$
2) if $\bar{y}$ is any solution of $(10.1.3)$, then there exists $\bar{c} \in \mathbb{R}$ such that the function on right of $(10.1.5)$ is equal to $\bar{y}$.

We prove 1). Let y be the function $(10.1.5)$, where c is any real number.

Obviously

$$y' = -ca(x)e^{-\int_{x_0}^{x} a(t)\,dt}$$

$$-a(x)e^{-\int_{x_0}^{x} a(t)\,dt} \int_{x_0}^{x} \left[f(t)\, e^{\int_{x_0}^{t} a(z)\,dz} \right] dt + e^{-\int_{x_0}^{x} a(t)\,dt} f(x)\, e^{\int_{x_0}^{x} a(z)\,dz}.$$

Thus, it results

$$y' + a(x)y =$$

$$= -ca(x)e^{-\int_{x_0}^{x} a(t)\,dt} - a(x)e^{-\int_{x_0}^{x} a(t)\,dt} \int_{x_0}^{x} \left[f(t)\, e^{\int_{x_0}^{t} a(z)\,dz} \right] dt$$

$$+e^{-\int_{x_0}^{x} a(t)\,dt} f(x)\, e^{\int_{x_0}^{x} a(z)\,dz} + a(x) \cdot ce^{-\int_{x_0}^{x} a(t)\,dt}$$

$$+a(x) \cdot \left(e^{-\int_{x_0}^{x} a(t)\,dt} \int_{x_0}^{x} \left[f(t)\, e^{\int_{x_0}^{t} a(z)\,dz} \right] dt \right)$$

$$= e^{-\int_{x_0}^{x} a(t)\,dt} f(x)\, e^{\int_{x_0}^{x} a(z)\,dz} = 1 \cdot f(x) = f(x)$$

i.e. y is solution of $(10.1.3)$.
We prove 2). Let $\bar{y}$ be any solution of $(10.1.3)$, *i.e.* any particular integral of $(10.1.3)$. Hence $\forall x \in \Omega$

(10.1.6) $$\bar{y}' + a(x)\bar{y} = f(x).$$

We have seen by 1) that, if $c \in \mathbb{R}$, the function y given by the $(10.1.5)$ is solution of $(10.1.3)$ and then is such that

(10.1.7) $$y' + a(x)y = f(x).$$

Hence the function $y_1 = \bar{y} - y$ is such that $\forall x \in \Omega$

$$y_1' + a(x)y_1 = 0 \,, \tag{10.1.8}$$

i.e. it is solution of the associated homogeneous differential equation.
From $(10.1.8)$ we obtain

$$\frac{y_1'}{y_1} = -a(x) \,,$$

hence

$$\frac{d}{dx}\log y_1 = -a(x) \,,$$

hence the function $-a(x)$ admits the primitive $\log y_1$ and the primitive $\int_{x_0}^{x} -a(t)dt$, hence there exists $c_1 \in \mathbb{R}$ such that

$$\log y_1 = -\int_{x_0}^{x} a(t)dt + c_1 \,,$$

hence

$$y_1 = e^{c_1}\, e^{-\int_{x_0}^{x} a(t)dt}$$

hence

$$\bar{y} - y = y_1 = e^{c_1}\, e^{-\int_{x_0}^{x} a(t)dt}$$

hence

$$\bar{y} = y + e^{c_1}\, e^{-\int_{x_0}^{x} a(t)dt}$$

$$= ce^{-\int_{x_0}^{x} a(t)\, dt} + e^{-\int_{x_0}^{x} a(t)\, dt}\int_{x_0}^{x}\left[f(t)\, e^{\int_{x_0}^{t} a(z)\, dz}\right]dt + e^{c_1}\, e^{-\int_{x_0}^{x} a(t)dt}$$

hence the real number $\bar{c} = c + e^{c_1}$ is such that

$$\bar{y} = \bar{c}e^{-\int_{x_0}^{x} a(t)\,dt} + e^{-\int_{x_0}^{x} a(t)\,dt} \int_{x_0}^{x} \left[f(t)\, e^{\int_{x_0}^{t} a(z)\,dz} \right] dt\,. \ \diamond$$

Theorem 10.1.2 *Let*

- Ω *be an open interval of* $\mathbb{R}$
- $a : \Omega \to \mathbb{R}$
- *a be continuous*
- $x_0 \in \Omega$.

Then

(10.1.9) $$y = ce^{-\int_{x_0}^{x} a(t)\,dt}$$

is the general integral of the homogeneous differential equation (10.1.4).

Proof. It is the particular case $f = 0$ of theorem 10.1.1. ◊

Remark 10.1.4 Let

- Ω be an open interval of $\mathbb{R}$
- $p : \Omega \to \mathbb{R}$
- $q : \Omega \to \mathbb{R}$
- *p, q* be continuous
- $m \in \mathbb{R}$.

The first order differential equation on Ω

(10.1.10) $$y' + p(x)y + q(x)y^m = 0\,,$$

called *Bernouilli equation*, is not linear, but it can turn to a linear one.
In fact, from (10.1.10) it follows

$$y'y^{-m} + p(x)y^{1-m} + q(x)y^m = 0$$

and then, putting

$$u = y^{1-m}$$

we have

$$u' = (1-m)y^m y'.$$

So, the (10.1.10) turns to linear first order differential equation

$$\frac{u'}{1-m} + p(x)u + q(x) = 0$$

to which we can apply theorem 10.1.1. ⋄

Remark 10.1.5 Let

- Ω be an open interval of $\mathbb{R}$
- $p : \Omega \to \mathbb{R}$
- $q : \Omega \to \mathbb{R}$
- $r : \Omega \to \mathbb{R}$
- *p, q, r* be continuous.

The first order differential equation on Ω

$$y' + p(x)y + q(x)y^2 + r(x) = 0\,, \tag{10.1.11}$$

called *Riccati's equation*, is not linear. If a solution is known, the (10.1.11) can turn to a linear one.
In fact, let y_1 be such that

(10.1.12) $$y_1' + p(x)y_1 + q(x)y_1^2 + r(x) = 0\,.$$

We put

$$u = \frac{1}{y - y_1}\,,$$

hence

$$y = y_1 + \frac{1}{u}\,,$$

hence

$$y' = y_1' - \frac{u'}{u^2}\,,$$

hence the $(10.1.11)$ turns to

$$y_1' - \frac{u'}{u^2} + p(x)\left(y_1 + \frac{1}{u}\right) + q(x)\left(y_1^2 + \frac{1}{u^2} + 2\frac{y_1}{u}\right) + r(x) = 0\,,$$

i.e. to

$$-\frac{u'}{u^2} + p(x)\frac{1}{u} + 2y_1 q(x)\frac{1}{u} + [y_1' + p(x)y_1 + q(x)y_1^2 + r(x)] = 0,$$

i.e., taking account of $(10.1.12)$, to

$$u' + [p(x) + 2y_1 q(x)]u + q(x) = 0\,,$$

that is a linear first order differential equation, to which we can apply theorem 10.1.1. ◊

10.2 Differential equations of order $n > 1$

10.2.1 Introduction

Definition 10.2.1 Let

$$y^{(n)} = F\left(x, y, y', \dots, y^{(n-1)}\right) \tag{10.2.1}$$

be, on any open interval Ω of $\mathbb{R}$, an ordinary differential equation of order n in normal form. We call *problem with boundary conditions*, or *problem with initial conditions* the problem of find functions y satisfying

- $(10.2.1)$ on Ω
- assigned conditions on the boundary $\partial\Omega$ of Ω or at one or more points of Ω . ◇

Remark 10.2.1 The most important problem with initial conditions is the following *problem of Cauchy*:

- *Find a real function y defined in Ω , provided in Ω of derivatives at least up to n-th order and such that*

$$\begin{aligned}
&y^{(n)} = F\left(x, y, y', \dots, y^{(n-1)}\right) \qquad on\ \Omega \\
&y(x_0) = z_0 \in \mathbb{R} \\
&y'(x_0) = z_1 \in \mathbb{R} \\
&\ \dots \\
&y^{(n-1)}(x_0) = z_{n-1} \in \mathbb{R} \ ,
\end{aligned}$$

where $x_0 \in \Omega$.

In convenable hypotheses on F , a theorem states the existence and the uniqueness of the solution of the *Cauchy*'s problem. ◇

10.2.2 Linear differential equations

Definition 10.2.2 Let

- $n \in \mathbb{N}$
- Ω be any open interval of $\mathbb{R}$
- $a_1 \,:\, \Omega \to \mathbb{R}$
- ...
- $a_n \,:\, \Omega \to \mathbb{R}$
- $f \,:\, \Omega \to \mathbb{R}$.

We call *linear ordinary differential equation of n-th order* any differential equation on Ω of type

$$(10.2.2) \qquad y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \ . \ \diamond$$

Theorem 10.2.1 *Let*

- $n \in \mathbb{N}$
- Ω *be any open interval of* $\mathbb{R}$
- $a_1 \,:\, \Omega \to \mathbb{R}$
- ...
- $a_n \,:\, \Omega \to \mathbb{R}$
- $f \,:\, \Omega \to \mathbb{R}$
- $a_1, \dots, a_n, f$ *be continuous*
- $x_0 \in \Omega$
- $z_0, z_1, \dots, z_{n-1} \in \mathbb{R}$.

Then the problem (of Cauchy) to find a real function y *defined in* Ω *, provided in* Ω *of derivatives at least up to* n*-th order and such that*

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = f(x) \qquad \text{on } \Omega$$