

Systems of linear equations [◇]

2.1 Matrices

2.1.1 Vector spaces

Definition 2.1.1 Let X be a nonempty set and K be a field. We say that X is a *vector (or linear) space over the field K* if:

- X is provided with a binary operation, called *addition*, that maps any $(\mathbf{x}, \mathbf{y}) \in X \times X$ into one and only one element of X , denoted $\mathbf{x} + \mathbf{y}$, satisfying the following statements

$$(2.1.1) \quad \forall (\mathbf{x}, \mathbf{y}) \in X \times X \quad \mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$$

$$(2.1.2) \quad \forall (\mathbf{x}, \mathbf{y}, \mathbf{z}) \in X^3 \quad \mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$$

$$(2.1.3) \quad \exists \mathbf{0} \in X \text{ (called } \mathbf{zero}) \text{ such that } \forall \mathbf{x} \in X \quad \mathbf{0} + \mathbf{x} = \mathbf{x}$$

$$(2.1.4) \quad \forall \mathbf{x} \in X \quad \exists -\mathbf{x} \in X \text{ (called } \mathbf{opposite} \text{ of } \mathbf{x}) \text{ such that } \mathbf{x} + (-\mathbf{x}) = \mathbf{0};$$

- X is provided with an operation \cdot , called *scalar multiplication*, that maps any $(\alpha, \mathbf{x}) \in K \times X$ into one and only one element of X , called product of α and \mathbf{x} , denoted $\alpha \cdot \mathbf{x}$ (or $\alpha\mathbf{x}$), satisfying the following statements

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$$(2.1.5) \quad \forall (\alpha, \beta, \mathbf{x}) \in K \times K \times X \quad \alpha \cdot (\beta \cdot \mathbf{x}) = (\alpha \cdot \beta) \cdot \mathbf{x}$$

$$(2.1.6) \quad X \text{ contains an element } \mathbf{1} \neq \mathbf{0}, \text{ called } \textit{identity element}, \\ \text{such that } \forall \mathbf{x} \in X \quad \mathbf{1} \cdot \mathbf{x} = \mathbf{x};$$

- the operations of addition and multiplication obey the *distributive laws*

$$(2.1.7) \quad \forall (\alpha, \beta, \mathbf{x}) \in K \times K \times X \\ (\alpha + \beta) \cdot \mathbf{x} = \alpha \cdot \mathbf{x} + \beta \cdot \mathbf{x}$$

$$(2.1.8) \quad \forall (\alpha, \mathbf{x}, \mathbf{y}) \in K \times X \times X \\ \alpha \cdot (\mathbf{x} + \mathbf{y}) = \alpha \cdot \mathbf{x} + \alpha \cdot \mathbf{y} .$$

The elements of X are called *vectors* (or *points*), while the elements of K are called *scalars*. If $K = \mathbb{R}$, X is called *real vector* (or *linear*) *space*. \diamond

A very important example of vector space over a field is the *vector space* \mathbb{R}^n *over the real field*. Precisely, for each positive integer n , let \mathbb{R}^n be the set of all ordered n -tuples

$$\mathbf{x} = (x_1, x_2, \dots, x_n),$$

where x_1, x_2, \dots, x_n are real numbers. For each $\mathbf{y} = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$ and for each $\alpha \in \mathbb{R}$, we put

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n) \\ \alpha \mathbf{x} = (\alpha x_1, \dots, \alpha x_n)$$

so that $\mathbf{x} + \mathbf{y} \in \mathbb{R}^n$ and $\alpha \mathbf{x} \in \mathbb{R}^n$. We easily verify that such operations (addition of vectors and multiplication of a vector by a

scalar) satisfy the commutative, associative and distributive laws, as well as that the zero element of \mathbb{R}^n (sometimes called the *origin* or the *null vector*) is the point $\mathbf{0}$, all of whose coordinates are 0. Even the existence of the opposite element and the identity element is easily verified.

Definition 2.1.2 Let

$$\begin{aligned} \mathbf{x}_1 &= (x_{1,1}, x_{1,2}, \dots, x_{1,n}) \\ \mathbf{x}_2 &= (x_{2,1}, x_{2,2}, \dots, x_{2,n}) \\ &\dots \\ \mathbf{x}_k &= (x_{k,1}, x_{k,2}, \dots, x_{k,n}) \end{aligned}$$

be $k \in \mathbb{N}$ vectors of \mathbb{R}^n , and c_1, c_2, \dots, c_k be k real numbers. The vector of \mathbb{R}^n

$$(2.1.9) \quad \mathbf{x} = c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2 + \dots + c_k \mathbf{x}_k$$

is called the *linear combination* of the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ according the *coefficients* c_1, c_2, \dots, c_k . \diamond

Remark 2.1.1 Obviously, with the usual notation $\mathbf{x} = (x_1, x_2, \dots, x_n)$ the vector equality (2.1.9) is equivalent to system of k scalar equalities

$$(2.1.10) \quad \begin{aligned} x_1 &= c_1 x_{1,1} + c_2 x_{2,1} + \dots + c_k x_{k,1} \\ x_2 &= c_1 x_{1,2} + c_2 x_{2,2} + \dots + c_k x_{k,2} \\ &\dots \\ x_n &= c_1 x_{1,n} + c_2 x_{2,n} + \dots + c_k x_{k,n} . \diamond \end{aligned}$$

Remark 2.1.2 Obviously, the linear combination of any vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ according the coefficients $0, 0, \dots, 0$ is the null vector. \diamond

Definition 2.1.3 Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ be $k \in \mathbb{N}$ vectors of \mathbb{R}^n . If

$$(2.1.11) \quad c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_k\mathbf{x}_k = \mathbf{0}$$

only happens when $c_1 = c_2 = \dots = c_k = 0$, then the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are called *linearly independent*. \diamond

Definition 2.1.4 Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ be $k \in \mathbb{N}$ vectors of \mathbb{R}^n . If there exists a $(c_1, c_2, \dots, c_k) \in \mathbb{R}^k - \{\mathbf{0}\}$ such that

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \dots + c_k\mathbf{x}_k = \mathbf{0} ,$$

then the vectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ are called *linearly dependent*. \diamond

Remark 2.1.3 Clearly, the null vector is linearly dependent and every nonnull vector is linearly independent. \diamond

Theorem 2.1.1 Let X be a real vector space and $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_k$ ($k \in \mathbb{N}$) be a system of vectors of X . The following statements are equivalent:

- 1) the vectors are linearly dependent
- 2) one vector is a linear combination of the others.

Proof. 1) \Rightarrow 2). Let $c_1 \neq 0$ in (2.1.11). Then

$$x_1 = -\frac{c_2}{c_1} x_2 - \dots - \frac{c_k}{c_1} x_k .$$

2) \Rightarrow 1). Suppose that one of the vectors x_1, x_2, \dots, x_k ($k \in \mathbb{N}$) is a linear combination of the others. Then the vectors x_1, x_2, \dots, x_k are linearly dependent. \diamond

Definition 2.1.5 The real vector space X is called finite-dimensional and the number n is called the dimension of the space if there exist n linearly independent vectors in X , while any $n + 1$ vectors in X are linearly dependent. If the space contains linearly independent systems of an arbitrary number of vectors, then it is called infinite-dimensional. \diamond

Definition 2.1.6 Let X be a real vector space. We call *basis* of X a system of vectors of X if

- the vectors of basis are linearly independent
- the vector space X consists of all linear combinations of the vectors of basis. \diamond

Theorem 2.1.2 *The coordinate vectors*

$$(2.1.12) \quad \begin{aligned} \mathbf{e}_1 &= (1, 0, \dots, 0) \\ \mathbf{e}_2 &= (0, 1, \dots, 0) \\ &\dots \\ \mathbf{e}_n &= (0, 0, \dots, 1) \end{aligned}$$

are a basis of the real vector space \mathbb{R}^n .

Proof. Since the vector equation $c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0$ implies the scalar equations

$$\begin{aligned} c_1 &= 0 \\ c_2 &= 0 \\ &\dots \\ c_n &= 0, \end{aligned}$$

the coordinate vectors e_1, e_2, \dots, e_n are linearly independent.

Let $x = (x_1, x_2, \dots, x_n)$ be any vector of \mathbb{R}^n . Since

$$x = x_1e_1 + x_2e_2 + \cdots + x_ne_n,$$

the thesis is true. \diamond

Remark 2.1.4 There is one and only one way to write a vector as a linear combination of the basis vectors. In fact, if

$$\begin{aligned} x &= x_1e_1 + x_2e_2 + \cdots + x_ne_n \\ x &= c_1e_1 + c_2e_2 + \cdots + c_ne_n, \end{aligned}$$

then subtraction gives

$$0 = (x_1 - c_1)e_1 + (x_2 - c_2)e_2 + \cdots + (x_n - c_n)e_n.$$

By independence, every coefficient must be zero, hence $x_1 = c_1, x_2 = c_2, \dots, x_n = c_n$. \diamond

Remark 2.1.5 If $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$, then

$x = (x_1, x_2, \dots, x_n)$ and the numbers x_1, x_2, \dots, x_n are called the *coordinates of x in the basis e_1, e_2, \dots, e_n* . \diamond

2.1.2 Basics of matrices

Definition 2.1.7 Let $m, n \in \mathbb{N}$. A rectangular array of real numbers

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

is called a *real matrix*. When $m = n$, the matrix is called *square* and the number m , equal to n , is called its *order*. In the general case the matrix is called *rectangular*, of *dimension $m \times n$* . The numbers that constitute the matrix are called the *elements*. In the double-subscript notation for the elements, the first subscript always denotes the row and the second subscript the column containing the given element. \diamond

Definition 2.1.8 A rectangular matrix consisting of a single column

$$\begin{bmatrix} a_{1j} \\ \dots \\ a_{mj} \end{bmatrix}$$

is called *column matrix* or *column vector*. \diamond

Definition 2.1.9 A rectangular matrix consisting of a single

row

$$[a_{i1} \quad \dots \quad a_{in}]$$

is called *row matrix* or *row vector*. \diamond

Definition 2.1.10 A square matrix of order $n \in \mathbb{N}$, in which all the elements outside the main diagonal are zero

$$\begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix},$$

is called *diagonal matrix*. \diamond

Definition 2.1.11 A square matrix of order $n \in \mathbb{N}$, in which the main diagonal consists entirely of units and all the others elements are zero

$$U = \begin{bmatrix} 1 & \dots & 0 \\ \dots & \dots & \dots \\ 0 & \dots & 1 \end{bmatrix},$$

is called *unit matrix*. \diamond

Definition 2.1.12 A square matrix in which all the elements below the main diagonal are zero

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

is called *upper triangular*. \diamond

Definition 2.1.13 A square matrix in which all the elements above the main diagonal are zero

$$\begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ a_{21} & a_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

is called *lower triangular*. \diamond

Definition 2.1.14 Let us consider the rectangular matrix of dimension $m \times n$

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

We call *transpose* of A the matrix of dimension $n \times m$

$$A^T = \begin{bmatrix} a_{11}^T & \cdots & a_{1m}^T \\ \cdots & \cdots & \cdots \\ a_{n1}^T & \cdots & a_{nm}^T \end{bmatrix},$$

where $a_{ki}^T = a_{ik} \quad \forall i \in \{1, 2, \dots, m\}$ and $\forall k \in \{1, 2, \dots, n\}$.

\diamond

Definition 2.1.15 If a square matrix coincides with its transpose, then it is called *symmetric*. \diamond

Remark 2.1.6 Let A any matrix of dimension $m \times n$. The transpose matrix A^T has as first row the first column of A , ..., as n -th row the n -th column of A . \diamond

Definition 2.1.16 If a square matrix differs from its transpose by the factor -1 , then it is called *skew-symmetric*. \diamond

Remark 2.1.7 In any symmetric matrix elements that are symmetrically placed with respect to the main diagonal are equal. In a skew-symmetric matrix elements any two elements that are symmetrically placed with respect to the main diagonal differ each other by the factor -1 and the diagonal elements are zero. \diamond

2.1.3 Operations on matrices

Definition 2.1.17 Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, \quad B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}$$

be two rectangular matrices, both of dimension $m \times n$. We call *sum* (or *addition*) of A and B , and denote $A + B$, the matrix, of the same dimension, whose elements are the sums of the corresponding elements of the given matrices:

$$A + B = \begin{bmatrix} a_{11} + b_{11} & \dots & a_{1n} + b_{1n} \\ \dots & \dots & \dots \\ a_{m1} + b_{m1} & \dots & a_{mn} + b_{mn} \end{bmatrix}. \diamond$$

Remark 2.1.8 According to definition 2.1.17, only rectangular matrices of equal dimension can be added. \diamond

Remark 2.1.9 From the definition 2.1.17 it follows immediately that the matrix addition has the properties of commutativity ($A + B = B + A$) and of associativity ($(A + B) + C = A + (B + C)$). \diamond

Definition 2.1.18 Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

be a rectangular matrix, of dimension $m \times n$. We call *product* (or *multiplication*) of A by $\alpha \in \mathbb{R}$, and denote αA , the matrix, of the same dimension, whose elements are obtained from the corresponding elements of A by multiplication by α :

$$\alpha A = \begin{bmatrix} \alpha a_{11} & \dots & \alpha a_{1n} \\ \dots & \dots & \dots \\ \alpha a_{m1} & \dots & \alpha a_{mn} \end{bmatrix}. \diamond$$

Remark 2.1.10 Let A, B be two rectangular matrices of equal dimension. It is easy to see that $\forall \alpha, \beta \in \mathbb{R}$

$$(2.1.13) \quad \alpha(A + B) = \alpha A + \alpha B$$

$$(2.1.14) \quad (\alpha + \beta)A = \alpha A + \beta A$$

$$(2.1.15) \quad (\alpha \beta)A = \alpha(\beta A) . \diamond$$

Definition 2.1.19 Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & \dots & b_{1n} \\ \dots & \dots & \dots \\ b_{m1} & \dots & b_{mn} \end{bmatrix}$$

be two rectangular matrices, both of dimension $m \times n$. We call *difference* of A and B , and denote $A - B$, the matrix, of the same dimension

$$A - B = A + (-1)B . \diamond$$

Definition 2.1.20 Let

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \dots & \dots & \dots \\ a_{m1} & \dots & a_{mn} \end{bmatrix}$$

be a rectangular matrix of dimension $m \times n$,

$$B = \begin{bmatrix} b_{11} & \dots & b_{1q} \\ \dots & \dots & \dots \\ b_{n1} & \dots & b_{nq} \end{bmatrix}$$

be a rectangular matrix of dimension $n \times q$. We call *product* (or *multiplication*) of A and B , and denote AB , the matrix, of dimension $m \times q$

$$A B = \begin{bmatrix} c_{11} & \dots & c_{1q} \\ \dots & \dots & \dots \\ c_{m1} & \dots & c_{mq} \end{bmatrix}$$

in which the element c_{ij} at the intersection of the i -th row and the j -th column is the sum of the products of the corresponding elements (of the i -th row and the j -th column):

$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

$\forall i \in \{1, \dots, m\}$ and $\forall j \in \{1, \dots, q\}$. \diamond

Remark 2.1.11 According to definition 2.1.20, the operation of multiplication of two rectangular matrices can only be carried out when the number of columns of the first factor is equal to the number of rows of the second. \diamond

Remark 2.1.12 It is easy to see that the matrix multiplication has the following properties

$$(2.1.16) \quad (A B)C = A(B C)$$

$$(2.1.17) \quad (A + B)C = A C + B C$$

$$(2.1.18) \quad A(B + C) = A B + A C . \diamond$$

Theorem 2.1.3 *If a rectangular matrix A (of dimension $m \times n$) is multiplied on the right by a diagonal matrix $\{d_1, d_2, \dots, d_n\}$, then the columns of A are multiplied by d_1, d_2, \dots, d_n , respectively.*

Proof. In fact

$$\begin{aligned} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \\ &= \begin{bmatrix} a_{11}d_1 & a_{12}d_2 & \cdots & a_{1n}d_n \\ a_{21}d_1 & a_{22}d_2 & \cdots & a_{2n}d_n \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}d_1 & a_{m2}d_2 & \cdots & a_{mn}d_n \end{bmatrix} . \diamond \end{aligned}$$

Theorem 2.1.4 *If a rectangular matrix A (of dimension $m \times n$) is multiplied on the left by a diagonal matrix $\{d_1, d_2, \dots, d_m\}$, then the rows of A are multiplied by d_1, d_2, \dots, d_m , respectively.*

Proof. In fact

$$\begin{aligned} & \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & d_m \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \\ &= \begin{bmatrix} d_1a_{11} & d_1a_{12} & \cdots & d_1a_{1n} \\ d_2a_{21} & d_2a_{22} & \cdots & d_2a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ d_ma_{m1} & d_ma_{m2} & \cdots & d_ma_{mn} \end{bmatrix} . \diamond \end{aligned}$$

Theorem 2.1.5 *Let U be a unit matrix of order $n \in \mathbb{N}$.*

Then for every square matrix A of order $n \in \mathbb{N}$ we have

$$U A = A U = A .$$

Proof. Obvious. \diamond

Theorem 2.1.6 Let A, B be two rectangular matrices and $\alpha \in \mathbb{R}$. We have

$$(2.1.19) \quad (A + B)^T = A^T + B^T$$

$$(2.1.20) \quad (\alpha A)^T = \alpha A^T$$

$$(2.1.21) \quad (A B)^T = B^T A^T .$$

Proof. Obvious. \diamond

Definition 2.1.21 Let A any non-zero square matrix of order $n \in \mathbb{N}$, U the unit matrix of order n . We say that A is *invertible* if there exists a square matrix B of order n such that

$$A B = B A = U .$$

If A is invertible, B is called the *inverse matrix* of A and denoted A^{-1} . \diamond

Theorem 2.1.7 Let A be an invertible matrix of order $n \in \mathbb{N}$. The inverse of A is unique.

Proof. Let B, C two inverse matrices of A . Hence $A B = B A = U$ and $A C = C A = U$. Hence $C = C U = C(A B) = (C A)B = U B = B$. \diamond

Theorem 2.1.8 Let A, B be two invertible matrices of order $n \in \mathbb{N}$. The matrix AB is invertible and we have

$$(2.1.22) \quad (AB)^{-1} = B^{-1}A^{-1}.$$

Proof. We have $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = AA^{-1} = U$ and $(B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B = B^{-1}B = U$. From theorem 2.1.7 we have $(AB)^{-1} = B^{-1}A^{-1}$. \diamond

2.2 Determinants

2.2.1 Combinatorial analysis

Definition 2.2.1 Let $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$. Let A be the set constituted by n distinct elements a_1, a_2, \dots, a_n . We call k -combination of the set A any subset of k elements of A . The order of selection does not matter. \diamond

Definition 2.2.2 Let $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$. Let A be the set constituted by n distinct elements a_1, a_2, \dots, a_n . We call k -disposition of the set A any ordered subset of k elements of A . Different orderings correspond to different dispositions. \diamond

Definition 2.2.3 Let $n \in \mathbb{N}$ and A be the set constituted by n distinct elements a_1, a_2, \dots, a_n . We call *permutation* of the set A any ordering of the elements of A . \diamond

Remark 2.2.1 Notice that there's only one n -combination of the set $\{a_1, a_2, \dots, a_n\}$. \diamond

Remark 2.2.2 Notice that the n -dispositions of the set $\{a_1, a_2, \dots, a_n\}$ are exactly the *permutations* of the same set. \diamond

Definition 2.2.4 Let

- $n \in \mathbb{N}$
- $k \in \{1, \dots, n\}$
- $\{a_1, a_2, \dots, a_n\}$ be a set constituted by n distinct elements.

We denote

- $C_{n,k}$ the number of the k -combinations of $\{a_1, a_2, \dots, a_n\}$
- $D_{n,k}$ the number of the k -dispositions of $\{a_1, a_2, \dots, a_n\}$
- P_n the number of the *permutations* of $\{a_1, a_2, \dots, a_n\}$. \diamond

Theorem 2.2.1 Let us consider $n \in \mathbb{N}$, $k \in \{1, \dots, n\}$. Let $\{a_1, a_2, \dots, a_n\}$ be a set constituted by n distinct elements. It results

$$(2.2.1) \quad D_{n,k} = \frac{n!}{(n-k)!}$$

$$(2.2.2) \quad P_n = n!$$

$$(2.2.3) \quad C_{n,k} = \binom{n}{k}.$$

Proof. To calculate $D_{n,k}$, we distribute the k -dispositions of $\{a_1, a_2, \dots, a_n\}$ in n subsets. We put in the first subset the dispositions that have in the first place the element a_1 , in the

second the dispositions that have in the first place the element a_2, \dots , in the n -th the dispositions that have in the first place the element a_n . By symmetry, all such subsets have the same number of elements. Thus, $D_{n,k}$ is given by n multiplied by the number of elements presents in one subset. Obviously, such number is $D_{n-1,k-1}$. So, we have obtained the result

$$(2.2.4) \quad D_{n,k} = n D_{n-1,k-1} .$$

From the (2.2.4) and from the hypothesis $k \leq n$, we immediately obtain

$$\begin{aligned} D_{n-1,k-1} &= (n-1) D_{n-2,k-2} \\ D_{n-2,k-2} &= (n-2) D_{n-3,k-3} \\ &\dots \\ D_{n-(k-2),2} &= (n-(k-2)) D_{n-(k-1),k-(k-1)}, \end{aligned}$$

that is

$$\begin{aligned} D_{n-1,k-1} &= (n-1) D_{n-2,k-2} \\ D_{n-2,k-2} &= (n-2) D_{n-3,k-3} \\ &\dots \\ D_{n-k+2,2} &= (n-k+2) D_{n-k+1,1} . \end{aligned}$$

Obviously $D_{n-k+1,1} = n-k+1$. As a consequence, it results

$$\begin{aligned} D_{n,k} &= n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+2) \cdot (n-k+1) \\ &= \frac{n \cdot (n-1) \cdot (n-2) \cdot \dots \cdot (n-k+2) \cdot (n-k+1) \cdot (n-k)!}{(n-k)!} \end{aligned}$$

$$= \frac{n!}{(n-k)!} .$$

Thus, the (2.2.1) is proven.

The (2.2.2) immediately follows from the (2.2.1). In fact, we have

$$P_n = D_{n,n} = \frac{n!}{(n-n)!} = \frac{n!}{0!} = \frac{n!}{1} = n! .$$

About the (2.2.3), let's consider any k -combination of $\{a_1, a_2, \dots, a_n\}$. From this, by permutating in all possible ways its k elements, you will get $P_k = k!$. Thus, $D_{n,k} = k! C_{n,k}$. Hence

$$C_{n,k} = \frac{D_{n,k}}{k!} = \frac{n!}{k!(n-k)!} = \binom{n}{k} . \diamond$$

Definition 2.2.5 Let $n \in \mathbb{N}$. Let A be the set constituted by n distinct elements a_1, a_2, \dots, a_n . The ordering a_1, a_2, \dots, a_n is called *the fundamental permutation*. The generic permutation is denoted $a_{i_1}, a_{i_2}, \dots, a_{i_n}$, where $i_1, i_2, \dots, i_n \in \{1, 2, \dots, n\}$. In a generic permutation the place k is occupied by the element occupying the i_k place in the fundamental permutation. \diamond

Definition 2.2.6 Let $a_{i_1}, a_{i_2}, \dots, a_{i_n}$ be a generic permutation of the fundamental permutation a_1, a_2, \dots, a_n . Let $h, k \in \{1, 2, \dots, n\}$, $h < k$. We say that the elements a_{i_h}, a_{i_k} form an *inversion* if $i_h > i_k$. \diamond

To determine how many inversions appear in a permutation, you compare the first element with all those who follow it, then the second element with all those who follow it, and so on.

Definition 2.2.7 We say that a permutation has *even* [resp. *odd*] *class* if the overall number of its inversions is even [resp. odd]. \diamond

Theorem 2.2.2 *If in a permutation two elements are interchanged, then the permutation class changes.*

Proof. Let's examine first the case in which the two elements occupy in the permutation two consecutive places h and $h + 1$. The permutation can be represented, before the interchange by

$$(2.2.5) \quad A', a_h, a_{h+1}, A''$$

and after the interchange by

$$(2.2.6) \quad A', a_{h+1}, a_h, A'' .$$

Well, in both (2.2.5) and (2.2.6)

- every element of A' precedes both a_h and a_{h+1}
- every element of A'' follows both a_h and a_{h+1} .

So, if the (2.2.5) has m inversions, then the (2.2.6) has $m - 1$ or $m + 1$ inversions. Hence the two permutations have different class.