CHAPTER 7

INTEGRATION ◊

7.1 The definite integral

7.1.1 Definition of the definite integral

The *integration theory* was developed by *Riemann*[7.1.1] for real functions defined in intervals of real numbers. This theory is still useful and is presented in this chapter. Let's point out that it was extended from *Lebesgue*[7.1.2] to other functions, by a more general and more versatile theory.

Definition 7.1.1 Let $a, b \in \mathbb{R}, a < b$. By *partition* of the interval $[a, b]$ we mean a decomposition of $[a, b]$ obtained by $n \in \mathbb{N}$ points $x_1, \dots, x_n$, where

(7.1.1) $$a = x_0 < x_1 < \cdots < x_n = b\,. \diamond$$

Theorem 7.1.1 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$

◊ A. Maceri, *Integration*, e-ISBN 978-88-85929-70-8, © Accademica 2020

7.1.1 *Georg Friedrick Bernard Riemann*, Breselenz (Germany) 17.09.1826 – Selasca (Italy) 20.07.1866
7.1.2 *Henry Lebesgue*, Beauvais (France) – Paris 26.07.1941

- *f be continuous*
- $A = \left\{\sum_{i=1}^{n}\left[\left(\min_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right] \in \mathbb{R}\right.$ *such that* $n \in \mathbb{N}$
  and $x_1, \dots, x_n$ *is any partition of the interval* $\left.[a,b]\right\}$
- $B = \left\{\sum_{i=1}^{n}\left[\left(\max_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right] \in \mathbb{R}\right.$ *such that* $n \in \mathbb{N}$
  and $x_1, \dots, x_n$ *is any partition of the interval* $\left.[a,b]\right\}$.

Then the numerical sets A *and* B *are separate and contiguous* [7.1.3] .

Proof. To prove that the numerical sets A and B are separate we have to prove that

(7.1.2) $$(p \in A \text{ and } q \in B) \Rightarrow (p \le q) .$$

Let $p \in A, q \in B$. Hence there exists a partition $a = x_0 < x_1 < \cdots < x_m = b$ of $[a,b]$ such that

$$p = \sum_{i=1}^{m}\left[\left(\min_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right]$$

and there exists a partition $a = z_0 < z_1 < \cdots < z_n = b$ of $[a,b]$ such that

$$q = \sum_{j=1}^{n}\left[\left(\max_{z\in[z_{j-1},z_j]} f(z)\right)(z_j - z_{j-1})\right].$$

Obviously, the set

$$\{x_1, \dots, x_{m-1}\} \cup \{z_1, \dots, z_{n-1}\}$$

[7.1.3] See definition 1.2.4.

is a partition

$$a = y_0 < y_1 < \cdots < y_r = b$$

of $[a, b]$, where $r \in \mathbb{N}$ and $\forall i \in \{1, \dots, m\}$ the interval $[x_{i-1}, x_i]$ has the partition

$$x_{i-1} = y_{j_1} < \cdots < y_{j_i} = x_i \, .$$

This reasoning allows us to prove the (7.1.2). In fact, we have

$$p = \sum_{i=1}^{m} \left(\min_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1})$$
$$= \sum_{i=1}^{m} \left(\min_{x \in [x_{i-1}, x_i]} f(x) \right) \left(\sum_{k=j_1}^{j_i} (y_k - y_{k-1}) \right)$$
$$= \sum_{i=1}^{m} \sum_{k=j_1}^{j_i} \left[\left(\min_{x \in [x_{i-1}, x_i]} f(x) \right) (y_k - y_{k-1}) \right]$$
$$\leq \sum_{i=1}^{m} \sum_{k=j_1}^{j_i} \left[\left(\min_{x \in [y_{k-1}, y_k]} f(x) \right) (y_k - y_{k-1}) \right]$$
$$\leq \sum_{i=1}^{m} \sum_{k=j_1}^{j_i} \left[\left(\max_{x \in [y_{k-1}, y_k]} f(x) \right) (y_k - y_{k-1}) \right]$$
$$= \sum_{j=1}^{n} \sum_{h=i_1}^{i_j} \left[\left(\max_{x \in [y_{h-1}, y_h]} f(x) \right) (y_h - y_{h-1}) \right]$$
$$\leq \sum_{j=1}^{n} \sum_{h=i_1}^{i_j} \left[\left(\max_{x \in [z_{j-1}, z_j]} f(x) \right) (y_h - y_{h-1}) \right] =$$

$$= \sum_{j=1}^{n} \left[\left(\max_{z \in [z_{j-1}, z_j]} f(z) \right) (z_j - z_{j-1}) \right] = q \, .$$

Let us prove now that the numerical sets A and B are contiguous. We have to prove that

(7.1.3) $\forall \varepsilon > 0$ there exist $x \in A$ and $y \in B$ such that $y - x < \varepsilon$.

To this aim we consider, for every $n \in \mathbb{N}$

- the partition of $[a, b]$

$$x_0 = a$$
$$x_1 = x_0 + \frac{b-a}{n} > x_0$$
$$x_2 = x_1 + \frac{b-a}{n} = x_0 + 2\,\frac{b-a}{n} > x_1$$
$$x_3 = x_2 + \frac{b-a}{n} = x_0 + 3\,\frac{b-a}{n} > x_2$$
$$\dots$$
$$x_n = x_{n-1} + \frac{b-a}{n} = x_0 + n\,\frac{b-a}{n} = a + b - a = b > x_{n-1}$$

- the real numbers

(7.1.4) $$s_n = \sum_{i=1}^{n} \left[\left(\min_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right] \in A$$

(7.1.5) $$S_n = \sum_{i=1}^{n} \left[\left(\max_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right] \in B \, .$$

Clearly, to prove the (7.1.3) it is enough to prove that

$$\lim_{n\to+\infty}(S_n - s_n) = 0\,,$$

i.e. that

(7.1.6) $\qquad \forall\varepsilon > 0 \quad \exists\,\nu \in \mathbb{N} \quad : \quad \forall n > \nu \qquad |S_n - s_n| < \varepsilon\ .$

So, let $\varepsilon > 0$. By theorem 5.3.8 there exists $\delta > 0$ such that

(7.1.7) $\quad \forall x', x'' \in [a,b] \qquad (|x' - x''| < \delta)\left(|f(x') - f(x'')| < \dfrac{\varepsilon}{b-a}\right).$

We denote ν any positive integer such that $\nu > \frac{b-a}{\delta}$. As a consequence, for every $n > \nu$ we obviously have

(7.1.8) $$\frac{b-a}{n} < \delta\ .$$

On the other hand, by theorem 5.3.6 $\forall i \in \{1, \dots, n\}$ there exist $\xi_i, \eta_i \in [x_{i-1}, x_i] \subseteq [a,b]$ such that

(7.1.9) $\qquad f(\xi_i) = \max\limits_{x\in[x_{i-1},x_i]} f(x)\ , \quad f(\eta_i) = \min\limits_{x\in[x_{i-1},x_i]} f(x)\ .$

Since $x_{i-1} \le \xi_i \le x_i$ and $x_{i-1} \le \eta_i \le x_i$ we have $-(x_i - x_{i-1}) \le \xi_i - \eta_i \le x_i - x_{i-1}$ and then, taking into account the (7.1.8), $|\xi_i - \eta_i| \le |x_i - x_{i-1}| = x_i - x_{i-1} = \frac{b-a}{n} < \delta$. From this and from the (7.1.7) it follows $|f(\xi_i) - f(\eta_i)| < \frac{\varepsilon}{b-a}$. Consequently

$$|S_n - s_n| = \left|\sum_{i=1}^{n}\left[\left(\max_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right] - \sum_{i=1}^{n}\left[\left(\min_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right]\right|$$

$$= \left|\sum_{i=1}^{n}\left[\left(\max_{x\in[x_{i-1},x_i]} f(x) - \min_{x\in[x_{i-1},x_i]} f(x)\right)(x_i - x_{i-1})\right]\right|$$

$$< \left|\sum_{i=1}^{n}\left[\frac{\varepsilon}{b-a}(x_i - x_{i-1})\right]\right| = \left|\frac{\varepsilon}{b-a}\sum_{i=1}^{n}[(x_i - x_{i-1})]\right| = \varepsilon$$

and then the thesis is true. ◊

Remark 7.1.1 Theorem 7.1.1 examines the numerical sets A, B and proves thar they are separate and contiguous. So, by theorem 1.2.7, there exists one and only one $l \in \mathbb{R}$ (called *separation element*) such that

(7.1.10) $$a \le l \le b \qquad \forall a \in A \text{ and } \forall b \in B\ .$$ ◊

Remark 7.1.2 The proof of theorem 7.1.1 shows that the numerical sequences $\{s_n\}$ and $\{S_n\}$ are both convergent towards the same limit. Since the (7.1.10), such limit is the separation element l , *i.e.*

(7.1.11) $$\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} s_n = l\ .$$ ◊

Definition 7.1.2 Let

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- f continuous.

The separation element $(7.1.10)$ is a real number called *Riemann integral* (or *definite integral*) of f over $[a,b]$ and is denoted

(7.1.12) $$\int_a^b f(x)\,dx \;. \diamond$$

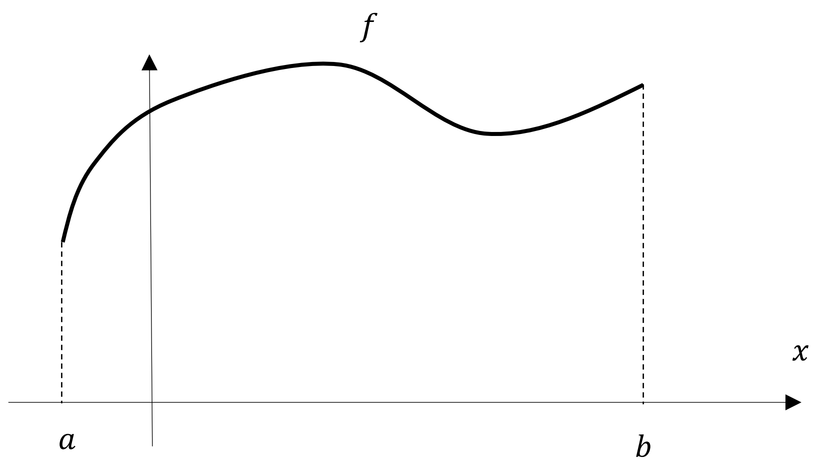

Fig. 7.1.1

Remark 7.1.3 The definite integral (7.1.12) has a *geometric interpretation,* very important for the applications.
Let $a,b \in \mathbb{R}, a < b, f$ be a function that maps $[a,b]$ into $\mathbb{R}$. Let O,x,y be, in a plane, a system of orthogonal *Cartesian* axes. Consider, for each $x \in X$, the point P of the plane having abscissa x and ordinate $y = f(x)$. As P varies in X, the point $P = (x, f(x))$ describes in the plane a set of points called *Cartesian diagram of the function* f.
We denote S the part of plane bounded by the *Cartesian* diagram of the function f, by the x-axis and by the straight lines $x = a$ and $x = b$ (fig. 7.1.1).

Let us consider now

- any partition $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$
- the multirectangle (contained in S) constituted by the union of the rectangles with base $[x_0, x_1], [x_1, x_2], \dots, [x_1, x_2]$ and, respectively, height $\min_{x \in [x_0, x_1]} f(x)$, $\min_{x \in [x_1, x_2]} f(x)$, ..., $\min_{x \in [x_1, x_2]} f(x)$. The area of such multirectangle contained in S is, obviously, given by $s_n = \sum_{i=1}^{n} \left[\left(\min_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right] \in \mathbb{R}$ (fig. 7.1.2)
- the multirectangle (containing S) constituted by the union of the rectangles with base $[x_0, x_1], [x_1, x_2], \dots, [x_1, x_2]$ and, respectively, height $\max_{x \in [x_0, x_1]} f(x)$, $\max_{x \in [x_1, x_2]} f(x)$, ..., $\max_{x \in [x_1, x_2]} f(x)$. The area of such multirectangle containing S is, obviously, given by $S_n = \sum_{i=1}^{n} \left[\left(\max_{x \in [x_{i-1}, x_i]} f(x) \right) (x_i - x_{i-1}) \right] \in \mathbb{R}$ (fig. 7.1.3).

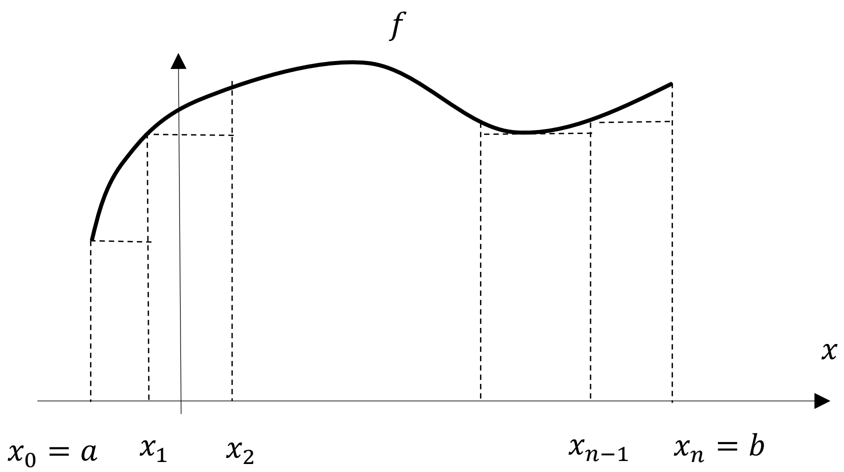

Fig. 7.1.2

The (7.1.10) and (7.1.11) clearly show that

- the definite integral (7.1.12) measure the area of S

- such value can be calculated, in a simple manner but with the wanted approximation, by the (7.1.11). ⋄

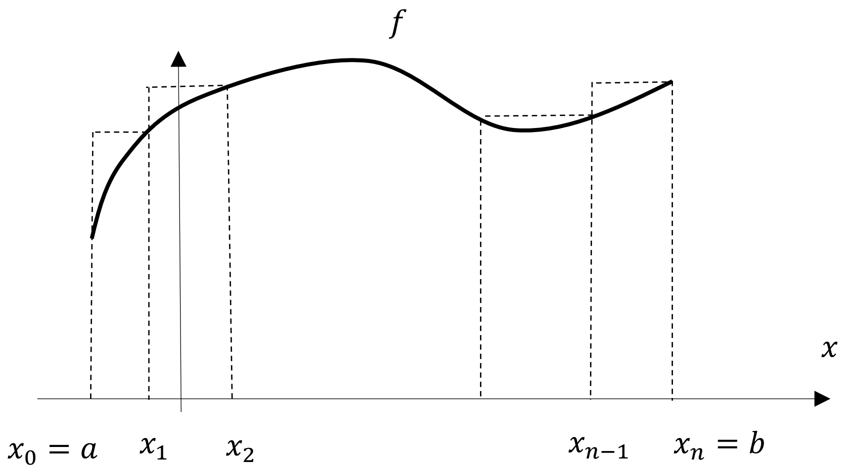

Fig. 7.1.3

Remark 7.1.4 In the (7.1.12) x is usually called *variable of integration*. We underline that it is immaterial which letter we use. So, we can indicate the definite integral, without distinction, with one of the symbols

$$\int_a^b f(x)\,dx\,, \quad \int_a^b f(y)\,dy\,, \quad \int_a^b f(z)\,dz\;. \quad \diamond$$

Remark 7.1.5 The (7.1.10), (7.1.11), (7.1.4), (7.1.5) banally imply

$$(7.1.13) \qquad \forall n \in \mathbb{N} \qquad s_n \le \int_a^b f(x)\,dx \le S_n$$

$$(7.1.14) \qquad \int_a^b f(x)\,dx = \lim_{n\to+\infty} S_n = \lim_{n\to+\infty} s_n \;. \quad \diamond$$

Remark 7.1.6 Let us consider the numerical sequences $\{s_n\}$ and $\{S_n\}$ respectively given by (7.1.4) and (7.1.5). For every $i \in \{1, \dots, n\}$ let us denote ρ_i any point of $[x_{i-1}, x_i]$. Hence $\forall i \in \{1, \dots, n\}$ it results

$$\min_{x\in[x_{i-1},x_i]} f(x) \le f(\rho_i) \le \max_{x\in[x_{i-1},x_i]} f(x) \;,$$

hence $\forall n \in \mathbb{N}$ it results

$$(7.1.15) \qquad s_n \le \sum_{i=1}^{n} [f(\rho_i)\cdot(x_i - x_{i-1})] \le S_n \;.$$

The $(7.1.15), (7.1.14)$ and theorem 4.1.10 imply

$$(7.1.16) \qquad \int_a^b f(x)\,dx = \lim_{n\to+\infty} \sum_{i=1}^{n} [f(\rho_i)\cdot(x_i - x_{i-1})] \;. \quad \diamond$$

7.1.2 Properties of the definite integral

Theorem 7.1.2 *Let*

- $a, b, k \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- $f(x) = k \qquad \forall x \in [a, b]$.

Then it results

(7.1.17) $$\int_a^b k\,dx = k(b-a)\,.$$

Proof. From (7.1.4) and (7.1.5) we immediately obtain

$$\forall n \in \mathbb{N} \qquad s_n = S_n = k(b-a)\,.$$

This result and the (7.1.11) imply the (7.1.17) . ◊

Remark 7.1.7 Theorem 7.1.2 banally implies

(7.1.18) $$\int_a^b 0\,dx = 0\,.$$ ◊

Theorem 7.1.3 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a,b] \to \mathbb{R}$
- f *be continuous*
- $f(x) \geq 0 \qquad \forall x \in [a,b]$.

Then it results

(7.1.19) $$\int_a^b f(x)\,dx \geq 0\,.$$

Proof. From (7.1.4) we immediately obtain

$$\forall n \in \mathbb{N} \qquad s_n \geq 0.$$

This result, the (7.1.13) and theorem 4.1.9 imply the (7.1.19) . ◇

Theorem 7.1.4 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f_1 : [a, b] \to \mathbb{R}$
- $f_2 : [a, b] \to \mathbb{R}$
- f_1 *and* f_2 *be continuous*
- $c_1, c_2 \in \mathbb{R}$.

Then it results

$$(7.1.20) \qquad \int_a^b (c_1 f_1 + c_2 f_2)(x)\, dx = c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx\, .$$

Proof. From (7.1.16) we obtain

$$\int_a^b (c_1 f_1 + c_2 f_2)(x)\, dx = \lim_{n \to +\infty} \sum_{i=1}^{n} [(c_1 f_1 + c_2 f_2)(\rho_i) \cdot (x_i - x_{i-1})]$$

$$= \lim_{n \to +\infty} \left\{ c_1 \sum_{i=1}^{n} [f_1(\rho_i) \cdot (x_i - x_{i-1})] + c_2 \sum_{i=1}^{n} [f_2(\rho_i) \cdot (x_i - x_{i-1})] \right\}$$

$$= c_1 \lim_{n \to +\infty} \sum_{i=1}^{n} [f_1(\rho_i) \cdot (x_i - x_{i-1})] + c_2 \lim_{n \to +\infty} \sum_{i=1}^{n} [f_2(\rho_i) \cdot (x_i - x_{i-1})]$$

$$= c_1 \int_a^b f_1(x)\, dx + c_2 \int_a^b f_2(x)\, dx \ . \ \diamond$$

Theorem 7.1.5 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- $g: [a, b] \to \mathbb{R}$
- f *and* g *be continuous*
- $f(x) \leq g(x) \qquad \forall x \in [a, b]$.

Then it results

$$\int_a^b f(x)\, dx \leq \int_a^b g(x)\, dx \ . \tag{7.1.21}$$

Proof. By theorem 7.1.3 we have

$$\int_a^b [g(x) - f(x)]\, dx \geq 0 \ .$$

Consequently, by theorem 7.1.4 it results

$$\int_a^b g(x)\, dx - \int_a^b f(x)\, dx \geq 0$$

and this proves the (7.1.21). ◊

Theorem 7.1.6 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- f *be continuous.*

Then it results

$$\left| \int_a^b f(x)\, dx \right| \le \int_a^b |f(x)|\, dx \,. \tag{7.1.22}$$

Proof. Since the (1.2.17) we have

$$\forall x \in [a, b] \qquad -|f(x)| \le f(x) \le |f(x)| \,.$$

Consequently, from theorems 7.1.5 and 7.1.4, we have

$$-\int_a^b |f(x)|\, dx = \int_a^b -|f(x)|\, dx \le \int_a^b f(x)\, dx \le \int_a^b |f(x)|\, dx$$

and then, since the (1.2.18), the (7.1.22) is true. ◊

Definition 7.1.3 Let

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- f *be continuous.*

We put

(7.1.23) $$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$$

(7.1.24) $$\int_a^a f(x)\,dx = 0\ . \ \diamond$$

Theorem 7.1.7 *Let*

- $a, b, c \in \mathbb{R}$
- $a < b$
- $f: [min\{a, b, c\}, max\{a, b, c\}] \to \mathbb{R}$
- f *be continuous.*

Then it results

(7.1.25) $$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\ .$$

Proof. The possible cases are

1. $c \in\ ]a, b[$
2. $c \in \{a, b\}$
3. $c \notin [a, b]$.

In case 1 we execute the partition of $[a, c]$ and $[c, b]$ with, respectively

$$a = x_0 < x_1 < \cdots x_{h-1} < x_h = c$$
$$c = x_h < x_{h+1} < \cdots x_{n-1} < x_n = b\ .$$

By employing the (7.1.16) we easily obtain

$$\int_a^b f(x)\,dx = \lim_{n\to+\infty}\sum_{i=1}^{n}[f(\rho_i)\cdot(x_i - x_{i-1})]$$
$$= \lim_{n\to+\infty}\sum_{i=1}^{h}[f(\rho_i)\cdot(x_i - x_{i-1})] + \lim_{n\to+\infty}\sum_{i=h+1}^{n}[f(\rho_i)\cdot(x_i - x_{i-1})]$$
$$= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

In case 2 the (7.1.25) banally follows from (7.1.24).
In case 3 or $a < b < c$ or $c < a < b$. If $a < b < c$, the case 1 gives us

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx\,.$$

This result and the (7.1.23) imply the (7.1.25).
A similar reasoning can be applied to case $c < a < b$. ◊

Theorem 7.1.8 [*mean value*] *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f: [a, b] \to \mathbb{R}$
- f *be continuous.*

Then there exists $c \in [a, b]$ *such that*

(7.1.26) $$\int_a^b f(x)\,dx = f(c)\,(b - a)\,.$$

Proof. Because the *Weierstrass* theorem there exist $m, M \in \mathbb{R}$ such that $m = \min_{x \in [a,b]} f(x)$ and $M = \max_{x \in [a,b]} f(x)$. Hence

$$\forall x \in [a,b] \qquad m \leq f(x) \leq M$$

hence, taking account of theorems 7.1.5 and 7.1.2, we have

$$m(b-a) = \int_a^b m\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b M\,dx = M(b-a) \ ,$$

hence

$$m \leq \frac{\int_a^b f(x)\,dx}{b-a} \leq M \ . \tag{7.1.27}$$

By virtue of $(7.1.27)$ and theorem 5.3.3, there exists $c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = f(c)\,(b-a) \ . \ \diamond$$

Theorem 7.1.9 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a,b] \to \mathbb{R}$
- $g : [a,b] \to \mathbb{R}$
- f *and* g *be continuous*
- $g \geq 0$.

Then there exists $c \in [a,b]$ *such that*

$$(7.1.28) \qquad \int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx \ .$$

Proof. Putting $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$ we have

$$\forall x \in [a,b] \qquad m \le f(x) \le M$$

and then

$$\forall x \in [a,b] \qquad m\,g(x) \le f(x)\,g(x) \le M\,g(x)$$

and then

$$m\int_a^b g(x)\,dx \le \int_a^b f(x)g(x)\,dx \le M\int_a^b g(x)\,dx \ .$$

So, if $\int_a^b g(x)\,dx = 0$, we have $\int_a^b f(x)g(x)\,dx = 0$ and the $(7.1.28)$ is true. If $\int_a^b g(x)\,dx \neq 0$, we have

$$m \le \frac{\int_a^b f(x)g(x)\,dx}{\int_a^b g(x)\,dx} \le M$$

and the $(7.1.28)$ is true by virtue of theorem 5.3.3. ◊

Definition 7.1.4 Let $a,b \in \mathbb{R}$ such that $a < b$, $f : [a,b] \to \mathbb{R}$. We call *primitive* of f any function $G : [a,b] \to \mathbb{R}$ differentiable in $[a,b]$ and such that

(7.1.29) $\forall x \in [a,b] \quad G'(x) = f(x)$. ◊

Remark 7.1.8 Let $a, b \in \mathbb{R}$ such that $a < b$, $f : [a,b] \to \mathbb{R}$, G any primitive of f. We notice that $\forall c \in \mathbb{R}$ the function $G + c$ is a primitive of f. ◊

Theorem 7.1.10 *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a,b] \to \mathbb{R}$
- $F : [a,b] \to \mathbb{R}$
- $G : [a,b] \to \mathbb{R}$
- *F and G be primitive of f.*

Then there exists $c \in \mathbb{R}$ such that

(7.1.30) $\forall x \in [a,b] \quad F(x) = G(x) + c$.

Proof, The thesis immediately follows from theorem 6.1.14. ◊

Theorem 7.1.11 [*the fundamental theorem of integral calculus*] *Let*

- $a, b \in \mathbb{R}$
- $a < b$
- $f : [a,b] \to \mathbb{R}$
- *f be continuous.*

Then there exists a primitive of f. Furthermore, for any primitive G of f it results

(7.1.31) $$\int_a^b f(x)\,dx = G(b) - G(a) \ .$$

Proof. Let us consider the function (called *integral function*)

(7.1.32) $$F \ : \ x \in [a,b] \ \rightarrow \ F(x) = \int_a^x f(t)\,dt \in \mathbb{R} \, .$$

Let $x \in [a,b]$. We know that F is differentiable at x if there exists the limit

$$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h}$$

and it is a real number. Well, we have (by virtue of theorems 7.1.7 and 7.1.8)

(7.1.33) $$\lim_{h\to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h\to 0} \frac{\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt}{h}$$

$$= \lim_{h\to 0} \frac{\int_x^{x+h} f(t)\,dt}{h} = \lim_{h\to 0} f\big(l(h)\big)$$

and, since $l(h)$ is a point of the interval having as end points x and $x+h$

(7.1.34) $$\lim_{h\to 0} l(h) = x \, .$$

From $(7.1.33), (7.1.34)$ and theorem 5.1.23 we get