Chapter 9

Calculus by elementary functions ◊

9.1 Elementary functions

9.1.1 *Power function with integer exponent*

We recall that

- a real number x is called an *integer* if $x = 0$, or $x \in \mathbb{N}$, or $-x \in \mathbb{N}$
- the set of all integer is denoted by $\mathbb{Z}$.

Definition 9.1.1 Let $x \in \mathbb{R}$, $n \in \mathbb{Z}$. We call n*-th power of* x, and denote x^n, the real number

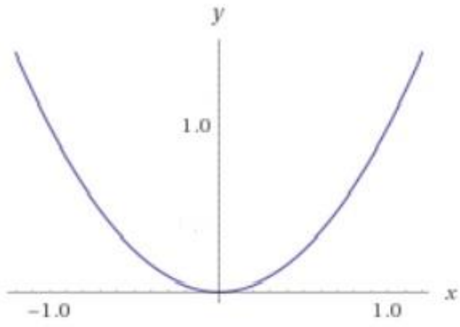
- $x^n = 1$, if $n = 0$
- $x^n = x$, if $n = 1$
- $x^n = x^{n-1}x$, if $n \in \mathbb{N} - \{1\}$
- $x^n = \frac{1}{x^{-n}}$, if $n < 0$ and $x \neq 0$.

The function x^n is called the *power function with integer exponent* n. Moreover

- if the integer n is positive and even, then the function x^n has *domain* $\mathbb{R}$ and *range* $[0, +\infty[$ (fig. 9.1.1)

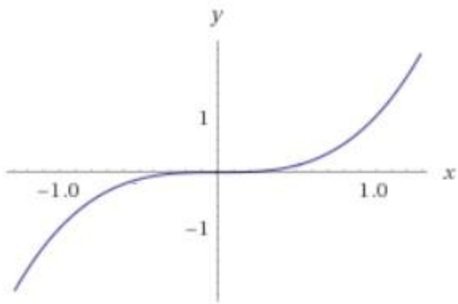
◊ A. Maceri, *Calculus by elementary functions*, e-ISBN 978-88-85929-72-2, © Accademica 2020

- if the integer n is positive and odd, then the function x^n has *domain* $\mathbb{R} - \{0\}$ and *range* $\mathbb{R}$ (fig. 9.1.2)



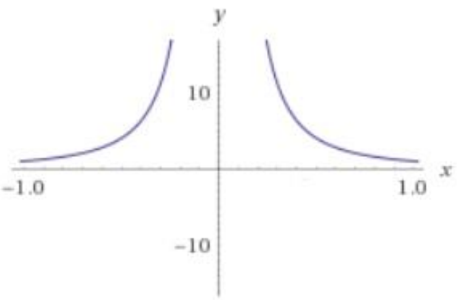
x^n (n even positive integer)

Fig. 9.1.1



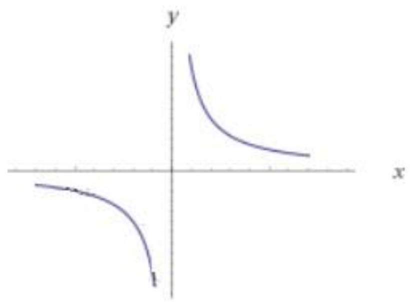
x^n (n odd positive integer)

Fig. 9.1.2



x^{-n} (n even positive integer)

Fig. 9.1.3



x^{-n} (n odd positive integer)

Fig. 9.1.4

- if the integer n is positive and even, then the function x^{-n} has *domain* $\mathbb{R}-\{0\}$ and *range* $]0\,,+\infty[$ (fig. 9.1.3)

- if the integer n is positive and odd, then the function x^{-n} has *domain* $\mathbb{R}-\{0\}$ and *range* $\mathbb{R}-\{0\}$ (fig.

9.1.4). ⋄

We recall that $\forall\, n\,, m \in \mathbb{N}\,,\ \forall\, x \in \mathbb{R}$ and $\forall\, y \in \mathbb{R} - \{0\}$

- $x^n\, x^m = x^{n+m}$
- $\frac{y^n}{y^m} = y^{n-m}$
- $(x^n)^m = x^{nm}$
- $(x\, y)^n = x^n\, y^n$
- $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.

Extrema:

- if the integer n is positive and even, then
  - ✓ $\min_{\mathbb{R}} x^n = 0$ (fig. 9.1.1)
  - ✓ $\sup_{\mathbb{R}} x^n = +\infty$ (fig. 9.1.1)
- if the integer n is positive and odd, then
  - ✓ $\inf_{\mathbb{R}} x^n = -\infty$ (fig. 9.1.2)
  - ✓ $\sup_{\mathbb{R}} x^n = +\infty$ (fig. 9.1.2)
- if the integer n is positive and even, then
  - ✓ $\inf_{\mathbb{R}-\{0\}} x^{-n} = 0$ (fig. 9.1.3)
  - ✓ $\sup_{\mathbb{R}-\{0\}} x^{-n} = +\infty$ (fig. 9.1.3)
- if the integer n is positive and odd, then

✓ $\inf_{\mathbb{R}-\{0\}} x^{-n} = -\infty$ (fig. 9.1.4)

✓ $\sup_{\mathbb{R}-\{0\}} x^{-n} = +\infty$ (fig. 9.1.4).

Monotonicity:

- if the integer n is positive and even, then the function x^n
  - ✓ is strictly increasing on $[0\,,+\infty[$ (fig. 9.1.1)
  - ✓ is strictly decreasing on $]-\infty\,,0]$ (fig. 9.1.1)

- if the integer n is positive and odd, then the function x^n is strictly increasing on $\mathbb{R}$ (fig. 9.1.2)

- if the integer n is positive and even, then the function x^{-n}
  - ✓ is strictly increasing on $]-\infty\,,0[$ (fig. 9.1.3)
  - ✓ is strictly decreasing on $]0\,,+\infty[$ (fig. 9.1.3)

- if the integer n is positive and odd, then the function x^{-n}
  - ✓ is strictly decreasing on $]-\infty\,,0[$ (fig. 9.1.4)
  - ✓ is strictly decreasing on $]0\,,+\infty[$ (fig. 9.1.4).

Limits:

- if the integer n is positive and even, then
  - ✓ $\lim_{x\to+\infty} x^n = +\infty$ (fig. 9.1.1)
  - ✓ $\lim_{x\to-\infty} x^n = +\infty$ (fig. 9.1.1)
- if the integer n is positive and odd, then
  - ✓ $\lim_{x\to+\infty} x^n = +\infty$ (fig. 9.1.2)
  - ✓ $\lim_{x\to-\infty} x^n = -\infty$ (fig. 9.1.2)
- if the integer n is positive and even, then
  - ✓ $\lim_{x\to+\infty} x^{-n} = 0$ and then the x-axis is horizontal asymptote on right (fig. 9.1.3)
  - ✓ $\lim_{x\to 0^+} x^{-n} = +\infty$ and then the y-axis is vertical asymptote on right (fig. 9.1.3)
  - ✓ $\lim_{x\to 0^-} x^{-n} = +\infty$ and then the y-axis is vertical asymptote on left (fig. 9.1.3)
  - ✓ $\lim_{x\to-\infty} x^{-n} = 0$ and then the x-axis is horizontal asymptote on left (fig. 9.1.3)

- if the integer n is positive and odd, then
  - ✓ $\lim_{x\to+\infty} x^{-n} = 0$ and then the x-axis is horizontal asymptote on right (fig. 9.1.4)

- ✓ $\lim_{x \to 0^+} x^{-n} = +\infty$ and then the y-axis is vertical asymptote on right (fig. 9.1.4)
- ✓ $\lim_{x \to 0^-} x^{-n} = -\infty$ and then the y-axis is vertical asymptote on left (fig. 9.1.4)
- ✓ $\lim_{x \to -\infty} x^{-n} = 0$ and then the x-axis is horizontal asymptote on left (fig. 9.1.4).

Continuity: the power function with integer exponent n is continuous.

Differentiability: for every $n \in \mathbb{Z}$ it results

(9.1.1) $$\frac{dx^n}{dx} = n\, x^{n-1} \ .$$

In fact, we obviously have

(9.1.2) $$\frac{dx}{dx} = \lim_{h \to 0} \frac{x + h - x}{h} = 1 \ .$$

Hence, if $n > 0$ we have

$$\frac{dx^n}{dx} = \frac{d}{dx}(x \cdot x \cdot \ldots \cdot x) = x^{n-1}\frac{dx}{dx} + \cdots + x^{n-1}\frac{dx}{dx} = n\, x^{n-1} \, .$$

Furthermore, we obviously have

(9.1.3) $$\frac{dx^{-1}}{dx} = \frac{d}{dx}\frac{1}{x} = \lim_{h \to 0} \frac{\dfrac{1}{x+h} - \dfrac{1}{x}}{h} = \lim_{h \to 0} \frac{-1}{x^2 + hx} = -\frac{1}{x^2} \ .$$

Hence, if $n > 0$ we have

$$\frac{dx^{-n}}{dx}=\frac{d}{dx}\left(\frac{1}{x}\cdot\frac{1}{x}\cdot\ldots\cdot\frac{1}{x}\right)=\frac{1}{x^{n-1}}\frac{d}{dx}\frac{1}{x}+\cdots+\frac{1}{x^{n-1}}\frac{d}{dx}\frac{1}{x}=-n\,x^{-n-1}\,.$$

Integrability: we immediately verify that, $\forall\, n\in\mathbb{Z}\,,$ a primitive of x^n is $\frac{x^{n+1}}{n+1}$.

9.1.2 *n-th Root function* ($n\in\mathbb{N}$)

Definition 9.1.2 Let $n\in\mathbb{N}$. The restriction of x^n to $[0\,,+\infty[$, *i.e.*, the function

$$x^n \;:\; x\in[0\,,+\infty[\;\rightarrow x^n\in[0\,,+\infty[\;,$$

is strictly increasing and then it possesses the inverse. We call the inverse *n-th root of* x, and denote it $\sqrt[n]{x}$. So, the function $\sqrt[n]{x}$ has *domain* $[0\,,+\infty[$, *range* $[0\,,+\infty[$ and it results

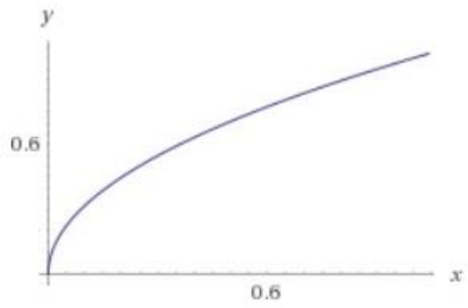
(9.1.4) $$\forall\, x\in[0\,,+\infty[\qquad \left(\sqrt[n]{x}\right)^n=x\ .$$

The (9.1.4) allows us to consider the operation of extraction of the n-*th* root of x identical to elevation of x to a power of exponent $\frac{1}{n}$. In fact, we have

$$\forall\, x\in[0\,,+\infty[\qquad \left(\sqrt[n]{x}\right)^n=\left(x^{\frac{1}{n}}\right)^n=x^{\frac{n}{n}}=x\,.$$

Thus, we write without distinction $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$. In fig. 9.1.5 we show the diagram of the function $\sqrt[n]{x}$ (*i.e.*, $x^{\frac{1}{n}}$). In case $n=$

2 , we write $\sqrt{x}$ instead of $\sqrt[2]{x}$. ◊



$\sqrt[n]{x}$ $(n \in \mathbb{N})$

Fig. 9.1.5

We recall that, $\forall\, n \in \mathbb{N}$ and $\forall\, x\,, z \in [0\,, +\infty[$, it results

$$\sqrt[n]{x}\ \sqrt[n]{z} = \sqrt[n]{x\,z}$$

$$\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}}\ .$$

Extrema: it results

✓ $\min\limits_{[0,+\infty[} \sqrt[n]{x} = 0$ (fig. 9.1.5)

✓ $\sup\limits_{[0,+\infty[} \sqrt[n]{x} = +\infty$ (fig. 9.1.5).

Monotonicity:

the function *n-th* root of x is strictly increasing (fig.

9.1.5)

Limits: it results (fig. 9.1.5)

$$\lim_{x \to +\infty} \sqrt[n]{x} = +\infty \ .$$

Continuity: the function n-*th* root of x is continuous.

Differentiability: we'll show in the following that for every $\alpha \in \mathbb{R}$ the power function

$$x^\alpha$$

is differentiable and it results

$$\frac{d}{dx} \, x^\alpha = \alpha \, x^{\alpha - 1} \, .$$

Hence

$$\frac{d}{dx} \sqrt[n]{x} = \frac{d}{dx} x^{\frac{1}{n}} = \frac{1}{n} x^{\frac{1}{n}-1} = \frac{1}{n} x^{-\frac{n-1}{n}} = \frac{1}{n\sqrt[n]{x^{n-1}}} \ . \tag{9.1.5}$$

In particular, we have

$$\frac{d}{dx} \sqrt{x} = \frac{1}{2\sqrt{x}} \, . \tag{9.1.6}$$

Integrability: we immediately verify that, $\forall \, n \in \mathbb{N}$, a primitive of $\sqrt[n]{x}$ is $\frac{n}{n+1} \sqrt[n]{x^{n+1}}$.

Remark 9.1.1 Let us notice that, if n is odd, since the power function x^n is strictly increasing on $\mathbb{R}$ and has range

$\mathbb{R}$, it also possesses a strictly increasing inverse $\sqrt[n]{x}$ with domain $\mathbb{R}$ and range $\mathbb{R}$. ⋄

9.1.3 *Exponential function*

Definition 9.1.3 Let $a \in]0,+\infty[$. If

- $x \in \mathbb{N}$, then a^x is defined as the product of n factors each of which is equal to a , *i.e.*, $a^x = a \cdot a \cdot \ldots \cdot a$
- $x = 0$, then a^x is defined as 1 , *i.e.*, $a^0 = 1$
- $x \in \mathbb{N}$, then a^{-x} is defined as $a^{-x} = \frac{1}{a^x}$
- $x \in \mathbb{N}$, then $a^{\frac{1}{x}}$ is defined as $a^{\frac{1}{x}} = \sqrt[x]{a}$
- $x \in \mathbb{N}$, then $a^{-\frac{1}{x}}$ is defined as $a^{-\frac{1}{x}} = \frac{1}{\sqrt[x]{a}}$
- $x = \frac{h}{k}$, where $h \in \mathbb{Z}$, $k \in \mathbb{Z} - \{0\}$ and $\frac{h}{k} > 0$, then a^x is defined as $a^x = a^{\frac{|h|}{|k|}} = \left(a^{|h|}\right)^{\frac{1}{|k|}} = \sqrt[|k|]{a^{|h|}}$
- $x = \frac{h}{k}$, where $h \in \mathbb{Z}$, $k \in \mathbb{Z} - \{0\}$ and $\frac{h}{k} < 0$, then a^x is defined as $a^x = a^{-\frac{|h|}{|k|}} = \left(a^{|h|}\right)^{-\frac{1}{|k|}} = \frac{1}{\sqrt[|k|]{a^{|h|}}} \sqrt[|k|]{a^{|h|}}$.

In this way we have defined a function a^x whose domain is $\mathbb{Q}$

(*i.e.*, the set of the rational numbers) and whose range is $]0\,,+\infty[$. Well, we can extend the function a^x to all irrational numbers so that the extended function, which we again denote a^x , is defined and continuous on $\mathbb{R}$. Moreover, the extended function possesses the property

$$\forall\, x\,,y \in \mathbb{R} \qquad a^{x+y} = a^x\, a^y\,.$$

We call the extended function a^x the *exponential function with base* a .

So, the domain of a^x is $\mathbb{R}$ and the range of a^x is $]0\,,+\infty[$. ◊

Remark 9.1.2 We put, for every $x \in \mathbb{R}$, $1^x = 1$. ◊

Extrema:

- $\inf\limits_{\mathbb{R}} a^x = 0$ (fig. 9.1.6 and fig. 9.1.7)
- $\sup\limits_{\mathbb{R}} a^x = +\infty$ (fig. 9.1.6 and fig. 9.1.7)

Monotonicity:

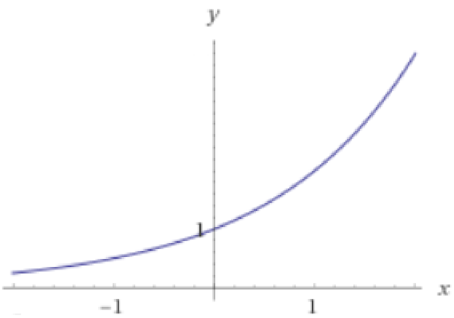
- If $a > 1$, then the function a^x is strictly increasing (fig. 9.1.6)

- If $0 < a < 1$, then the function a^x is strictly decreasing (fig. 9.1.7).

Monotonicity:

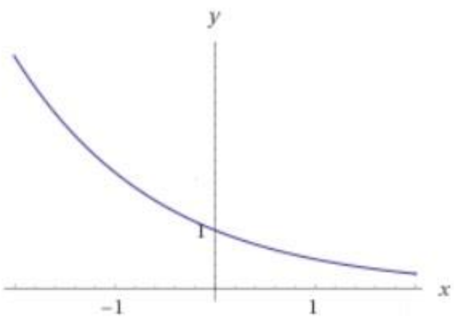
- If $a > 1$, then the function a^x is strictly increasing (fig. 9.1.6)

- If $0 < a < 1$, then the function a^x is strictly decreasing (fig. 9.1.7).



a^x $(a \in \mathbb{R}, a > 1)$

Fig. 9.1.6



a^x $(a \in \mathbb{R}, 0 < a < 1)$

Fig. 9.1.7

Limits:

- if $a > 1$, then

  - ✓ $\lim_{x\to+\infty} a^x = +\infty$ (fig. 9.1.6)
  - ✓ $\lim_{x\to-\infty} a^x = 0$ (fig. 9.1.6)

- if $0 < a < 1$, then

  - ✓ $\lim_{x\to+\infty} a^x = 0$ (fig. 9.1.7)
  - ✓ $\lim_{x\to-\infty} a^x = +\infty$ (fig. 9.1.7).

Continuity: the exponential function a^x is continuous.

Theorem 9.1.1 *Let* $a \in \,]0\,,+\infty[$. *It results*

(9.1.7)
$$\lim_{h\to 0}\frac{a^h - 1}{h} = \log a \ .$$

Proof. By theorem 4.1.31 we have

$$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^n = e \,,$$

hence

(9.1.8)
$$\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^n = \frac{\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)^{n+1}}{\lim_{n\to+\infty}\left(1+\frac{1}{n+1}\right)} = e \,,$$

hence

(9.1.9) $$\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n+1}$$

$$=\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)^{n}\cdot\lim_{n\to+\infty}\left(1+\frac{1}{n}\right)=e\,.$$

Putting $\forall\, x>0$

$$f_1(x)=\left(1+\frac{1}{[x]+1}\right)^{[x]}$$

$$f_2(x)=\left(1+\frac{1}{[x]}\right)^{[x]+1}$$

where $[x]$ denotes the greatest integer number smaller than x , from (9.1.8) we obtain

(9.1.10) $$\lim_{x\to+\infty}f_1(x)=e$$

and from (9.1.9) we obtain

(9.1.11) $$\lim_{x\to+\infty}f_2(x)=e\,.$$

Obviously $\forall\, x>0$

$$f_1(x)\le\left(1+\frac{1}{x}\right)^{x}\le f_2(x)$$

and then, taking account of (9.1.10), (9.1.11), it results

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^{x}=e$$

and consequently

(9.1.12) $$\lim_{y\to 0}(1+y)^{\frac{1}{y}} = e\ .$$

Let us consider now the function

$$\frac{a^x - 1}{x}$$

that is defined on $\mathbb{R}-\{0\}$ and at 0 it takes on the indeterminate form $\frac{0}{0}$. Putting

$$a^x - 1 = y$$

we have

$$x = \log_a(1+y)\ \ ,$$

hence when x tends to 0 then y tends to 0 and vice versa. Hence

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)} = \lim_{y\to 0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}}\ \ ,$$

hence, taking account of $(9.1.12)$, it results

(9.1.13) $$\lim_{x\to 0}\frac{a^x - 1}{x} = \frac{1}{\log_a e} = \log a\ .\ \diamond$$

Differentiability:

for every $a > 0$, taking account of $(9.1.13)$, it results

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h - 1}{h} = a^x\ \log a\ \ ,$$

hence

(9.1.14) $$\frac{d}{dx}e^x = e^x$$

and then $\forall\, n \in \mathbb{N}$

(9.1.15) $$\frac{d^n}{dx^n}e^x = e^x$$

and then, by the *Taylor*'s theorem 8.2.14, $\forall\, x \in \mathbb{R}$ it results

(9.1.16) $$e^x = \sum_{m=0}^{+\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \ .$$

Integrability:

we immediately verify that a primitive of e^x is e^x .

Remark 9.1.3 We underline that $\forall\, x\,, y \in \mathbb{R}$

- $e^x e^y = e^{x+y}$
- $e^0 = 1$
- $e^1 = e$
- $e^x > 0$
- $e^{-x} = \frac{1}{e^x}$. ◊

9.1.4 *Logarithm function*

Definition 9.1.4 Let $a \in \,]0\,,1[\, \cup \,]1\,,+\infty[$. The exponential function a^x is strictly monotone on $\mathbb{R}$ and then it possesses the inverse function. This function is called *the logarithm function to the base* a and is denoted

$$\log_a y \,.$$

Thus, the logarithm function to the base a has *domain* $]0\,,+\infty[$ and *range* $\mathbb{R}$. Moreover, there hold the identities

$$\forall\, x \in \mathbb{R} \qquad \log_a a^x = x$$

$$\forall\, x \in \,]0\,,+\infty[\qquad a^{\log_a x} = x \,.$$

If a is the *Napier* [9.1.1] number e, we prefer to write

$$\log x$$

rather than $\log_e x$. ◊

Remark 9.1.4 Let $a \in \,]0\,,1[\, \cup \,]1\,,+\infty[$. Banally, we have

$$\log_a 1 = 0 \,.$$

Moreover, we notice that $\forall\, x\,, y \in \,]0\,,+\infty[$ it results

$$a^{\log_a(xy)} = x\,y = a^{\log_a x}\, a^{\log_a y} = a^{\log_a x + \log_a y}$$

and then

(9.1.17) $$\log_a(x\,y) = \log_a x + \log_a y \,.$$

Replacing x by $\frac{x}{y}$ in (9.1.16) we obtain

(9.1.18) $$\log_a x - \log_a y = \log_a \frac{x}{y} \,.$$

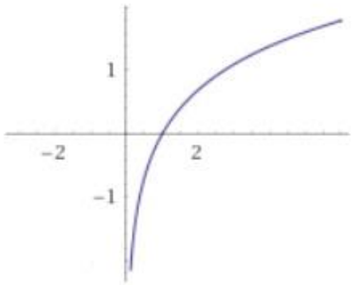
We also have $\forall\, x \in \,]0\,,+\infty[$

[9.1.1] *John Napier* , Edinburgh (Scotland) 01.02.1550 – Edinburgh 04.04.1617.

$$a^{\log_a x^y} = x^y = \left(a^{\log_a x}\right)^y = a^{y \log_a x}$$

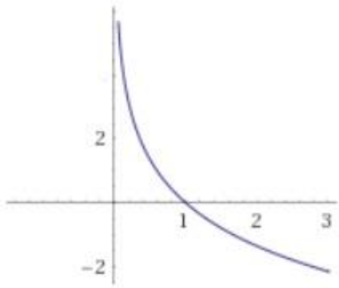
and therefore $\forall\, a \in \,]0\,,1[\, \cup \,]1\,,+\infty[$ and $\forall\, x \in \,]0\,,+\infty[$

(9.1.19) $$\log_a(x^y) = y \log_a x \;.$$



$\log_a x$ $(a \in \,]1,+\infty[)$

Fig. 9.1.8



$\log_a x$ $(a \in \,]0,1[)$

Fig. 9.1.9

Finally, $\forall\, a\,, b \in \,]0\,,1[\, \cup \,]1\,,+\infty[$ it results

$$a^{(\log_a b)\cdot(\log_b a)} = \left(a^{\log_a b}\right)^{\log_b a} = b^{\log_b a} = a \ ,$$

hence it results

(9.1.20) $$\log_a b = \frac{1}{\log_b a} \ . \ \diamond$$

Extrema:

- $\inf\limits_{]0,+\infty[} \log_a x = -\infty$ (fig. 9.1.8 and fig. 9.1.9)
- $\sup\limits_{]0,+\infty[} \log_a x = +\infty$ (fig. 9.1.8 and fig. 9.1.9).

Monotonicity:

- if $a > 1$, then the function $\log_a x$ is strictly increasing (fig. 9.1.8)
- if $0 < a < 1$, then the function $\log_a x$ is strictly decreasing (fig. 9.1.9).

Limits:

- if $a > 1$, then
  - ✓ $\lim\limits_{x \to +\infty} \log_a x = +\infty$ (fig. 9.1.8)
  - ✓ $\lim\limits_{x \to 0^+} \log_a x = -\infty$ and then the y-axis is vertical asymptote on right (fig. 9.1.8)
- if $0 < a < 1$, then

- ✓ $\lim_{x\to+\infty} \log_a x = -\infty$ (fig. 9.1.9)
- ✓ $\lim_{x\to 0^+} \log_a x = +\infty$ and then the y-axis is vertical asymptote on right (fig. 9.1.9).

Continuity: the logarithm function is continuous.

Differentiability: by theorem 6.1.7, putting $y = f(x)$ it results

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

i.e.,

$$\frac{d}{dy}\log_a y = \frac{1}{\frac{d}{dx}a^x} = \frac{1}{a^x \log a} = \frac{1}{y \log a}$$

i.e.,

(9.1.21) $$\frac{d}{dx}\log_a x = \frac{1}{x \log a} .$$

In particular, we have

(9.1.22) $$\frac{d}{dx}\log x = \frac{1}{x} .$$

Integrability: we immediately verify that a primitive of $\log x$ is $x\log x - x$.

9.1.5 *Power function with non-integer real exponent*

The power function x^α has already been defined

- in the case $\alpha \in \mathbb{Z}$ (definition 9.1.1). Precisely
  - ✓ if α is a positive integer the domain is $\mathbb{R}$
  - ✓ if α is a negative integer the domain is $\mathbb{R} - \{0\}$
- in the case $\alpha \in \mathbb{Q}$, where $\mathbb{Q}$ is the set of the rational numbers, (definition 9.1.1). The domain is $[0\,,+\infty[$.

Now we extend the definition to general case α non-integer real**.**

Definition 9.1.5 We define the *power function* x^α *with base* x *and exponent* α non-integer real

- if $x > 0$, to be the number $x^\alpha = e^{\log(x^\alpha)} = e^{\alpha \log x}$
- if $x = 0$ and $\alpha > 0$, to be the number $0^\alpha = 0$
- if $x = 0$ and $\alpha = 0$, to be the number $0^0 = 1$.

Thus

- if $\alpha > 1$, then x^α has *domain* $[0\,,+\infty[$ and *range* $[0\,,+\infty[$ (fig. 9.1.10)
- if $0 < \alpha < 1$, then x^α has *domain* $[0\,,+\infty[$ and *range* $[0\,,+\infty[$ (fig. 9.1.11)
- if $\alpha < 0$, then x^α has *domain* $]0\,,+\infty[$ and *range* $]0\,,+\infty[$ (fig. 9.1.12). ⋄

We recall that $\forall\, \alpha\,,\beta \in \mathbb{R}$ and $\forall\, x\,,y \in\,]0\,,+\infty[$

- $x^\alpha\, x^\beta = x^{\alpha+\beta}$