CHAPTER 9

CALCULUS BY ELEMENTARY FUNCTIONS ◊

9.1 Elementary functions

9.1.1 Power function with integer exponent

We recall that

- a real number x is called an *integer* if $x = 0$ or $x \in \mathbb{N}$ or $-x \in \mathbb{N}$
- the set of all integer is denoted by $\mathbb{Z}$.

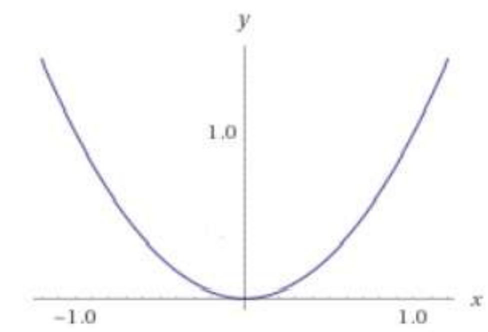
Definition 9.1.1 Let $x \in \mathbb{R}$, $n \in \mathbb{Z}$. We call n*-th power of* x , and denote x^n , the real number

- $x^n = 1$, if $n = 0$
- $x^n = x$, if $n = 1$
- $x^n = x^{n-1}x$, if $n \in \mathbb{N} - \{1\}$
- $x^n = \frac{1}{x^{-n}}$, if $n < 0$ and $x \neq 0$.

The function x^n is called the *power function with integer exponent* n . Moreover

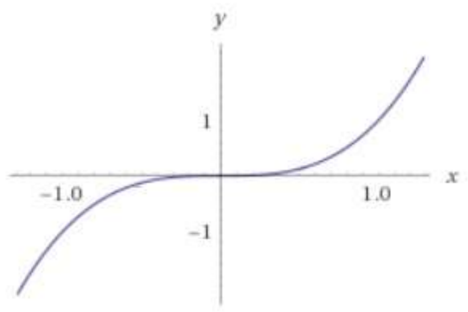
◊ A. Maceri, *Calculus by elementary functions*, e-ISBN 978-88-85929-72-2, © Accademica 2020

- if the integer n is positive and even, then the function x^n has *domain* $\mathbb{R}$ and *range* $[0, +\infty[$ (see fig. 9.1.1)
- if the integer n is positive and odd, then the function x^n has *domain* $\mathbb{R} - \{0\}$ and *range* $\mathbb{R}$ (see fig. 9.1.2)
- if the integer n is positive and even, then the function x^{-n} has *domain* $\mathbb{R} - \{0\}$ and *range* $]0, +\infty[$ (see fig. 9.1.3)
- if the integer n is positive and odd, then the function x^{-n} has *domain* $\mathbb{R} - \{0\}$ and *range* $\mathbb{R} - \{0\}$ (see fig. 9.1.4). ⋄



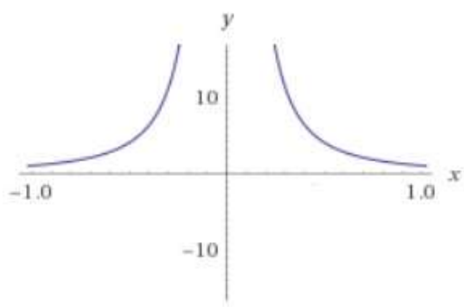
x^n (n even positive integer)

Fig. 9.1.1



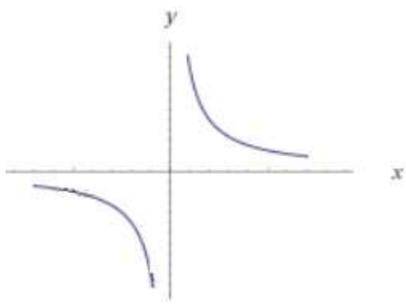
x^n (n odd positive integer)

Fig. 9.1.2



x^{-n} (n even positive integer)

Fig. 9.1.3



x^{-n} (n odd positive integer)

Fig. 9.1.4

We recall that $\forall n, m \in \mathbb{N}$, $\forall x \in \mathbb{R}$ and $\forall y \in \mathbb{R} - \{0\}$

- $x^n x^m = x^{n+m}$
- $\frac{y^n}{y^m} = y^{n-m}$
- $(x^n)^m = x^{nm}$
- $(xy)^n = x^n y^n$
- $\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$.

Extrema:

- if the integer n is positive and even, then
  - ✓ $\min_{\mathbb{R}} x^n = 0$ (see fig. 9.1.1)
  - ✓ $\sup_{\mathbb{R}} x^n = +\infty$ (see fig. 9.1.1)

- if the integer n is positive and odd, then
  - ✓ $\inf_{\mathbb{R}} x^n = -\infty$ (see fig. 9.1.2)
  - ✓ $\sup_{\mathbb{R}} x^n = +\infty$ (see fig. 9.1.2)

- if the integer n is positive and even, then
  - ✓ $\inf_{\mathbb{R}-\{0\}} x^{-n} = 0$ (see fig. 9.1.3)
  - ✓ $\sup_{\mathbb{R}-\{0\}} x^{-n} = +\infty$ (see fig. 9.1.3)

- if the integer n is positive and odd, then
  - ✓ $\inf_{\mathbb{R}-\{0\}} x^{-n} = -\infty$ (see fig. 9.1.4)
  - ✓ $\sup_{\mathbb{R}-\{0\}} x^{-n} = +\infty$ (see fig. 9.1.4).

Monotonicity:

- if the integer n is positive and even, then the function x^n
  - ✓ is strictly increasing on $[0, +\infty[$ (see fig. 9.1.1)
  - ✓ is strictly decreasing on $]-\infty, 0]$ (see fig. 9.1.1)

- if the integer n is positive and odd, then the function x^n
  - ✓ is strictly increasing on $\mathbb{R}$ (see fig. 9.1.2)

- if the integer n is positive and even, then the function x^{-n}
  - ✓ is strictly increasing on $]-\infty, 0[$ (see fig. 9.1.3)
  - ✓ is strictly decreasing on $]0, +\infty[$ (see fig. 9.1.3)

- if the integer n is positive and odd, then the function x^{-n}
  - ✓ is strictly decreasing on $]-\infty, 0[$ (see fig. 9.1.4)
  - ✓ is strictly decreasing on $]0, +\infty[$ (see fig. 9.1.4).

Limits:

- if the integer n is positive and even, then
  - ✓ $\lim_{x \to +\infty} x^n = +\infty$ (see fig. 9.1.1)
  - ✓ $\lim_{x \to -\infty} x^n = +\infty$ (see fig. 9.1.1)

- if the integer n is positive and odd, then
  - ✓ $\lim_{x \to +\infty} x^n = +\infty$ (see fig. 9.1.2)
  - ✓ $\lim_{x \to -\infty} x^n = -\infty$ (see fig. 9.1.2)

- if the integer n is positive and even, then
  - ✓ $\lim_{x \to +\infty} x^{-n} = 0$ and then the $x-$axis is horizontal asymptote on right (see fig. 9.1.3)
  - ✓ $\lim_{x \to 0^+} x^{-n} = +\infty$ and then the $y-$axis is vertical

asymptote on right (see fig. 9.1.3)
- ✓ $\lim_{x\to 0^-} x^{-n} = +\infty$ and then the $y-$axis is vertical asymptote on left (see fig. 9.1.3)
- ✓ $\lim_{x\to -\infty} x^{-n} = 0$ and then the $x-$axis is horizontal asymptote on left (see fig. 9.1.3)

- if the integer n is positive and odd, then
  - ✓ $\lim_{x\to +\infty} x^{-n} = 0$ and then the $x-$axis is horizontal asymptote on right (see fig. 9.1.4)
  - ✓ $\lim_{x\to 0^+} x^{-n} = +\infty$ and then the $y-$axis is vertical asymptote on right (see fig. 9.1.4)
  - ✓ $\lim_{x\to 0^-} x^{-n} = -\infty$ and then the $y-$axis is vertical asymptote on left (see fig. 9.1.4)
  - ✓ $\lim_{x\to -\infty} x^{-n} = 0$ and then the $x-$axis is horizontal asymptote on left (see fig. 9.1.4).

Continuity: the power function with integer exponent n is continuous.

Differentiability: for every $n \in \mathbb{Z}$ it results

(9.1.1) $$\frac{dx^n}{dx} = nx^{n-1} \ .$$

In fact, we obviously have

(9.1.2) $$\frac{dx}{dx} = \lim_{h\to 0}\frac{x+h-x}{h} = 1 \ .$$

Hence, if $n > 0$ we have

$$\frac{dx^n}{dx} = \frac{d}{dx}(x \cdot x \cdot \ldots \cdot x) = x^{n-1}\frac{dx}{dx} + \cdots + x^{n-1}\frac{dx}{dx} = nx^{n-1} \,.$$

Furthermore, we obviously have

$$\text{(9.1.3)} \qquad \frac{dx^{-1}}{dx} = \frac{d}{dx}\frac{1}{x} = \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h\to 0}\frac{-1}{x^2 + hx} = -\frac{1}{x^2} \,.$$

Hence, if $n > 0$ we have

$$\frac{dx^{-n}}{dx} = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{1}{x}\cdot \ldots \cdot\frac{1}{x}\right) = \frac{1}{x^{n-1}}\frac{d}{dx}\frac{1}{x} + \cdots + \frac{1}{x^{n-1}}\frac{d}{dx}\frac{1}{x} = -nx^{-n-1} \,.$$

Integrability: we immediately verify that, $\forall n \in \mathbb{Z}$, a primitive of x^n is $\frac{x^{n+1}}{n+1}$.

9.1.2 *n-th* Root function ($n \in \mathbb{N}$)

Definition 9.1.2 Let $n \in \mathbb{N}$. The restriction of x^n to $[0, +\infty[$, i.e. the function

$$x^n \;:\; x \in [0, +\infty[\,\to x^n \in [0, +\infty[\,,$$

is strictly increasing and then it possesses the inverse. We call the inverse *n-th root of x,* and denote it $\sqrt[n]{x}$. So, the function $\sqrt[n]{x}$ has *domain* $[0, +\infty[$, *range* $[0, +\infty[$ and it results

$$\text{(9.1.4)} \qquad \forall x \in [0, +\infty[\qquad \left(\sqrt[n]{x}\right)^n = x \,.$$

The $(9.1.4)$ allows us to consider the operation of extraction of th n-*th* root of x identical to elevation of x to a power of exponent $\frac{1}{n}$. In fact, we have

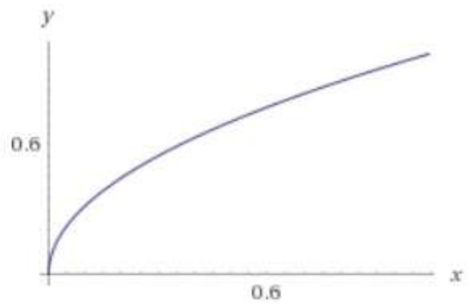
$$\forall x \in [0,+\infty[\qquad \left(\sqrt[n]{x}\right)^n = \left(x^{\frac{1}{n}}\right)^n = x^{\frac{n}{n}} = x \, .$$

Thus, we write without distinction $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$. In fig. 9.1.5 we show the diagram of the function $\sqrt[n]{x}$ (or $x^{\frac{1}{n}}$).
In case $n = 2$, we write $\sqrt{x}$ instead of $\sqrt[2]{x}$. ◊

We recall that, $\forall n \in \mathbb{N}$ and $\forall x, z \in [0,+\infty[$ it results

$$\sqrt[n]{x}\,\sqrt[n]{z} = \sqrt[n]{xz}$$

$$\frac{\sqrt[n]{x}}{\sqrt[n]{z}} = \sqrt[n]{\frac{x}{z}} \, .$$



$\sqrt[n]{x}$ $(n \in \mathbb{N})$

Fig. 9.1.5

Extrema: it results

- ✓ $\min\limits_{[0,+\infty[} \sqrt[n]{x} = 0$ (see fig. 9.1.5)
- ✓ $\sup\limits_{[0,+\infty[} \sqrt[n]{x} = +\infty$ (see fig. 9.1.5).

Monotonicity:

- the function n-*th* root of x is strictly increasing (see fig. 9.1.5)

Limits: it results

$$\lim_{x \to +\infty} \sqrt[n]{x} = +\infty \quad \text{(see fig. 9.1.5).}$$

Continuity: the function n-*th* root of x is continuous.

Differentiability: we'll show in the following that for every $\alpha \in \mathbb{R}$ the power function

$$x^\alpha$$

is differentiable and it results

$$\frac{d}{dx}\, x^\alpha = \alpha x^{\alpha-1} \,.$$

Hence

$$\text{(9.1.5)} \qquad \frac{d}{dx}\sqrt[n]{x} = \frac{d}{dx} x^{\frac{1}{n}} = \frac{1}{n} x^{\frac{1}{n}-1} = \frac{1}{n} x^{-\frac{n-1}{n}} = \frac{1}{n\sqrt[n]{x^{n-1}}} \,.$$

In particular, we have

$$\text{(9.1.6)} \qquad \frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}} \,.$$

Integrability: we immediately verify that, $\forall n \in \mathbb{N}$, a primitive of $\sqrt[n]{x}$ is $\frac{n}{n+1}\sqrt[n]{x^{n+1}}$.

Remark 9.1.1 Let us notice that, if n is odd, since the power function x^n is strictly increasing on $\mathbb{R}$ and has range $\mathbb{R}$, it also possesses a strictly increasing inverse $\sqrt[n]{x}$ with domain $\mathbb{R}$ and range $\mathbb{R}$. ◊

9.1.3 Exponential function

Definition 9.1.3 Let $a \in \,]0, +\infty[$. If

- $x \in \mathbb{N}$, then a^x is defined as the product of n factors each of which is equal to a , *i.e.* $a^x = a \cdot a \cdot \ldots \cdot a$
- $x = 0$, then a^x is defined as 1 , *i.e.* $a^x = 1$
- $x \in \mathbb{N}$, then a^{-x} is defined as $a^{-x} = \frac{1}{a^x}$
- $x \in \mathbb{N}$, then $a^{\frac{1}{x}}$ is defined as $a^{\frac{1}{x}} = \sqrt[x]{a}$
- $x \in \mathbb{N}$, then $a^{-\frac{1}{x}}$ is defined as $a^{-\frac{1}{x}} = \frac{1}{\sqrt[x]{a}}$
- $x = \frac{h}{k}$, where $h \in \mathbb{Z}$, $k \in \mathbb{Z} - \{0\}$ and $\frac{h}{k} > 0$, then a^x is defined as $a^x = a^{\frac{|h|}{|k|}} = \left(a^{|h|}\right)^{\frac{1}{|k|}} = \sqrt[|k|]{a^{|h|}}$
- $x = \frac{h}{k}$, where $h \in \mathbb{Z}$, $k \in \mathbb{Z} - \{0\}$ and $\frac{h}{k} < 0$, then a^x is defined as $a^x = a^{-\frac{|h|}{|k|}} = \left(a^{|h|}\right)^{-\frac{1}{|k|}} = \frac{1}{\sqrt[|k|]{a^{|h|}}} \sqrt[|k|]{a^{|h|}}$.

In this way we have defined a function a^x whose domain is $\mathbb{Q}$ (*i.e.* the set of the rational numbers) and whose range is $]0, +\infty[$.
Well, we can extend the function a^x to all irrational numbers so that the extended function, which we again denote a^x, is defined and continuous on $\mathbb{R}$. Moreover, the extended function possesses the property

$$\forall x, y \in \mathbb{R} \qquad a^{x+y} = a^x a^y \, .$$

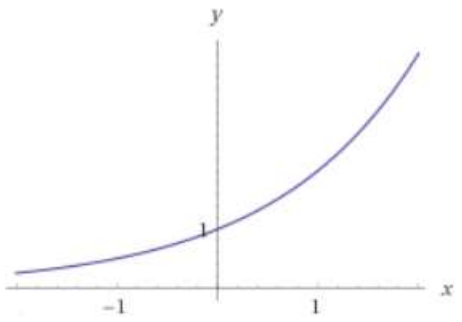
We call the extended function a^x the *exponential function with base* a .

So, the domain of a^x is $\mathbb{R}$ and the range of a^x is $]0,+\infty[$. ◊

Remark 9.1.2 We put, for every $x \in \mathbb{R}$, $1^x = 1$. ◊

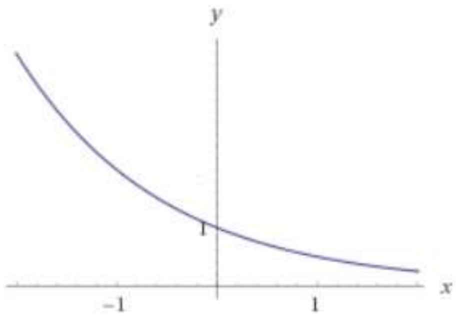
Extrema:

- $\inf\limits_{\mathbb{R}} a^x = 0$ (see fig. 9.1.6 and 9.1.7)
- $\sup\limits_{\mathbb{R}} a^x = +\infty$ (see fig. 9.1.6 and 9.1.7)



a^x $(a \in \mathbb{R}\, ,\, a > 1)$

Fig. 9.1.6



a^x $(a \in \mathbb{R}\, ,\, 0 < a < 1)$

Fig. 9.1.7

Monotonicity:

- if $a > 1$, then the function a^x is strictly increasing (see fig. 9.1.6)
- if $0 < a < 1$, then the function a^x is strictly decreasing (see fig. 9.1.7).

Limits:

- if $a > 1$, then
  - ✓ $\lim\limits_{x \to +\infty} a^x = +\infty$ (see fig. 9.1.6)

  - ✓ $\lim_{x \to -\infty} a^x = 0$ (see fig. 9.1.6)

- ○ if $0 < a < 1$, then
  - ✓ $\lim_{x \to +\infty} a^x = +\infty$ (see fig. 9.1.7)
  - ✓ $\lim_{x \to -\infty} a^x = +\infty$ (see fig. 9.1.7).

Continuity: the exponential function a^x is continuous.

Theorem 9.1.1 *Let* $a \in]0, +\infty[$. *It results*

$$\lim_{h \to 0} \frac{a^h - 1}{h} = \log a \ . \tag{9.1.7}$$

Proof. By theorem 4.1.31 we have

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n = e \,,$$

hence

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n+1}\right)^n = \frac{\lim_{n \to +\infty} \left(1 + \frac{1}{n+1}\right)^{n+1}}{\lim_{n \to +\infty} \left(1 + \frac{1}{n+1}\right)} = e \,, \tag{9.1.8}$$

hence

$$\lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^{n+1} = \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right)^n \cdot \lim_{n \to +\infty} \left(1 + \frac{1}{n}\right) = e \ . \tag{9.1.9}$$

Putting $\forall x > 0$

$$f_1(x) = \left(1 + \frac{1}{[x] + 1}\right)^{[x]}$$

$$f_2(x) = \left(1 + \frac{1}{[x]}\right)^{[x]+1}$$

where $[x]$ denotes the greatest integer number smaller than x , from $(9.1.8)$ we obtain

$$(9.1.10) \qquad \lim_{x \to +\infty} f_1(x) = e$$

and from $(9.1.9)$ we obtain

$$(9.1.11) \qquad \lim_{x \to +\infty} f_2(x) = e \, .$$

Obviously $\forall x > 0$

$$f_1(x) \leq \left(1 + \frac{1}{x}\right)^x \leq f_2(x)$$

and then, taking account of $(9.1.10), (9.1.11)$, it results

$$\lim_{x \to +\infty} \left(1 + \frac{1}{x}\right)^x = e$$

and consequently

$$(9.1.12) \qquad \lim_{y \to 0} (1 + y)^{\frac{1}{y}} = e \, .$$

Let us consider now the function

$$\frac{a^x - 1}{x}$$

that is defined on $\mathbb{R} - \{0\}$ and at 0 it takes on the indeterminate form $\frac{0}{0}$.

Putting

$$a^x - 1 = y$$

we have

$$x = \log_a(1+y) \ ,$$

hence when x tends to 0 then y tends to 0 and vice versa, hence

$$\lim_{x\to 0}\frac{a^x - 1}{x} = \lim_{y\to 0}\frac{y}{\log_a(1+y)} = \lim_{y\to 0}\frac{1}{\log_a(1+y)^{\frac{1}{y}}} ,$$

hence, taking account of $(9.1.12)$, it results

(9.1.13) $$\lim_{x\to 0}\frac{a^x - 1}{x} = \frac{1}{\log_a e} = \log a \ . \ \diamond$$

Differentiability: for every $a > 0$, taking account of $(9.1.13)$, it results

$$\frac{d}{dx}a^x = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = a^x \lim_{h\to 0}\frac{a^h - 1}{h} = a^x \log a \ ,$$

hence

(9.1.14) $$\frac{d}{dx}e^x = e^x$$

and then $\forall n \in \mathbb{N}$

(9.1.15) $$\frac{d^n}{dx^n}e^x = e^x$$

and then, by the *Taylor*'s theorem 8.2.14, $\forall x \in \mathbb{R}$ it results

(9.1.16) $$e^x = \sum_{m=0}^{+\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots \ .$$

Integrability:

- we immediately verify that a primitive of e^x is e^x .

Remark 9.1.3 We underline that $\forall x, y \in \mathbb{R}$

- $e^x e^y = e^{x+y}$
- $e^0 = 1$
- $e^1 = e$
- $e^x > 0$
- $e^{-x} = \frac{1}{e^x}$. ◊

9.1.4 Logarithm function

Definition 9.1.4 Let $a \in \,]0,1[\, \cup \,]1,+\infty[$ .The exponential function a^x is strictly monotone on $\mathbb{R}$ and then it possesses the inverse function. This function is called *the logarithm function to the base* a and is denoted

$$\log_a y \, .$$

Thus, the logarithm function to the base a has *domain* $]0,+\infty[$ and *range* $\mathbb{R}$. Moreover, there hold the identities

$$\forall x \in \mathbb{R} \qquad \log_a a^x = x$$

$$\forall x \in \,]0,+\infty[\qquad a^{\log_a x} = x \ .$$

If a is the *Napier* [9.1.1] number e, we prefer to write

$$\log x$$

rather than $\log_e x$. ◊

Remark 9.1.4 Let $a \in \,]0,1[\, \cup \,]1,+\infty[$. Banally, we have

$$\log_a 1 = 0 .$$

Moreover, we notice that $\forall x, y \in \,]0,+\infty[$ it results

$$a^{\log_a(xy)} = xy = a^{\log_a x} a^{\log_a y} = a^{\log_a x + \log_a y}$$

and then

(9.1.17) $$\log_a(xy) = \log_a x + \log_a y .$$

Replacing x by $\frac{x}{y}$ in (9.1.16) we obtain

(9.1.18) $$\log_a x - \log_a y = \log_a \frac{x}{y} .$$

We also have $\forall x \in \,]0,+\infty[$

$$a^{\log_a x^y} = x^y = \left(a^{\log_a x}\right)^y = a^{y \log_a x}$$

and therefore $\forall a \in \,]0,1[\, \cup \,]1,+\infty[$ and $\forall x \in \,]0,+\infty[$

(9.1.19) $$\log_a(x^y) = \ y\, log_a x \ .$$

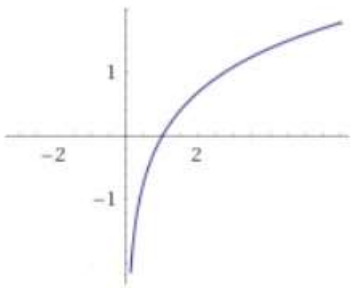
Finally, $\forall a, b \in \,]0,1[\, \cup \,]1,+\infty[$ it results

[9.1.1] *John Napier* , Edinburgh (Scotland) 01.02.1550 – Edinburgh 04.04.1617.

$$a^{(\log_a b)\cdot(\log_b a)} = \left(a^{\log_a b}\right)^{\log_b a} = b^{\log_b a} = a \ ,$$

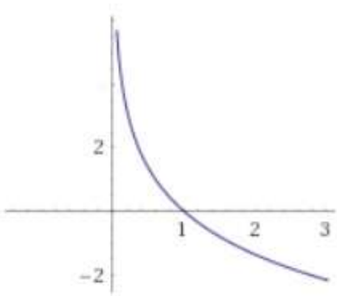
hence it results

(9.1.20) $$\log_a b = \frac{1}{\log_b a} \ .$$



$\log_a x \quad (a \in]1,+\infty[)$

Fig. 9.1.8



$\log_a x \quad (a \in]0,1[)$

Fig. 9.1.9

Extrema:

- $\inf\limits_{]0,+\infty[} \log_a x = -\infty$ (see fig. 9.1.8 and fig. 9.1.9)
- $\sup\limits_{]0,+\infty[} \log_a x = +\infty$ (see fig. 9.1.8 and fig. 9.1.9).

Monotonicity:

- if $a > 1$, then the function $\log_a x$ is strictly increasing (see fig. 9.1.8)
- if $0 < a < 1$, then the function $\log_a x$ is strictly decreasing (see fig. 9.1.9).

Limits:

- if $a > 1$, then
  - ✓ $\lim_{x\to+\infty} \log_a x = +\infty$ (see fig. 9.1.8)
  - ✓ $\lim_{x\to0^+} \log_a x = -\infty$ and then the $y-$axis is vertical asymptote on right (see fig. 9.1.8)

- if $0 < a < 1$, then
  - ✓ $\lim_{x\to+\infty} \log_a x = -\infty$ (see fig. 9.1.9)
  - ✓ $\lim_{x\to0^+} \log_a x = +\infty$ and then the $y-$axis is vertical asymptote on right (see fig. 9.1.9).

Continuity: the logarithm function is continuous.

Differentiability: by theorem 6.1.7, putting $y = f(x)$ it results

$$(f^{-1})'(y) = \frac{1}{f'(x)}$$

i.e.

$$\frac{d}{dy}\log_a y = \frac{1}{\frac{d}{dx}a^x} = \frac{1}{a^x \log a} = \frac{1}{y\log a}$$

i.e.

(9.1.21) $$\frac{d}{dx}\log_a x = \frac{1}{x\log a} \ .$$

In particular, we have

(9.1.22) $$\frac{d}{dx}\log x = \frac{1}{x} \ .$$

Integrability: we immediately verify that a primitive of $\log x$ is $x \log x - x$.

9.1.5 Power function with real exponent

The power function x^α has already been defined

- in the case $\alpha \in \mathbb{Z}$ (definition 9.1.1). Precisely
  - ✓ if α is a positive integer the domain is $\mathbb{R}$
  - ✓ if α is a negative integer the domain is $\mathbb{R} - \{0\}$
- in the case $\alpha \in \mathbb{Q}$, *i.e.* the set of the rational numbers, (definition 9.1.1). The domain is $[0, +\infty[$.

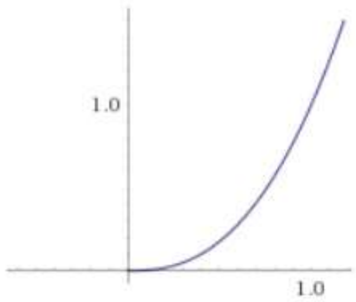
Now we extend the definition to case $\alpha \in \mathbb{R}$.

Definition 9.1.5 We define the *power function* x^α *with base* x *and exponent* $\alpha \in \mathbb{R}$

- if $x > 0$, to be the number $x^\alpha = e^{\log(x^\alpha)} = e^{\alpha \log x}$
- if $x = 0$ and $\alpha > 0,$ to be the number $0^\alpha = 0$
- if $x = 0$ and $\alpha = 0,$ to be (as usual) the number $0^0 = 1$.

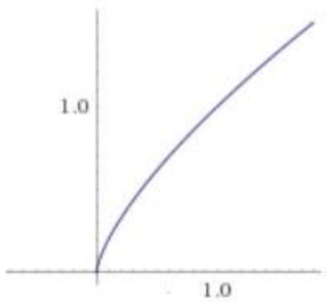
Thus

- if $\alpha > 1$, then x^α has *domain* $[0, +\infty[$ and *range* $[0, +\infty[$ (see fig. 9.1.10)
- if $0 < \alpha < 1$, then x^α has *domain* $[0, +\infty[$ and *range* $[0, +\infty[$ (see fig. 9.1.11)
- if $\alpha < 0$, then x^α has *domain* $]0, +\infty[$ and *range* $]0, +\infty[$ (see fig. 9.1.12). ⋄



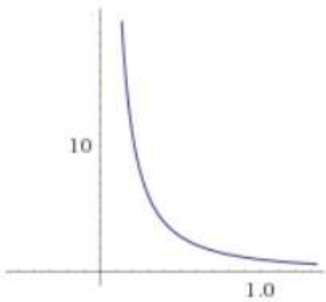
x^α ($\alpha \in]1, +\infty[$)

Fig. 9.1.10



x^α ($\alpha \in]0,1[$)

Fig. 9.1.11



x^α ($\alpha \in]-\infty, 0[$)

Fig. 9.1.12

We recall that $\forall \alpha, \beta \in \mathbb{R}$, $\forall x, y \in]0, +\infty[$

- $x^\alpha x^\beta = x^{\alpha+\beta}$
- $\frac{y^\alpha}{y^\beta} = y^{\alpha-\beta}$
- $(x^\alpha)^\beta = x^{\alpha\beta}$
- $(xy)^\alpha = x^\alpha y^\alpha$
- $\left(\frac{x}{y}\right)^\alpha = \frac{x^\alpha}{y^\alpha}$ . .

Extrema:

- if $\alpha \in]1,+\infty[$ then
  - ✓ $\min_{[0,+\infty[} x^\alpha = 0$ (see fig. 9.1.10)
  - ✓ $\sup_{[0,+\infty[} x^\alpha = +\infty$ (see fig. 9.1.10)
- if $\alpha \in]0,1[$ then
  - ✓ $\min_{[0,+\infty[} x^\alpha = 0$ (see fig. 9.1.11)
  - ✓ $\sup_{[0,+\infty[} x^\alpha = +\infty$ (see fig. 9.1.11)
- if $\alpha \in]-\infty,0[$ then
  - ✓ $\inf_{]0,+\infty[} x^\alpha = 0$ (see fig. 9.1.12)
  - ✓ $\sup_{]0,+\infty[} x^\alpha = +\infty$ (see fig. 9.1.12).

Monotonicity:

- if $\alpha \in]1,+\infty[$ then the function x^α is strictly increasing (see fig. 9.1.10)
- if $\alpha \in]0,1[$ then the function x^α is strictly increasing (see fig. 9.1.11)
- if $\alpha \in]-\infty,0[$ then the function x^α is strictly decreasing (see fig. 9.1.12)

Limits:

- if $\alpha \in]1,+\infty[$ then $\lim_{x\to+\infty} x^\alpha = +\infty$ (see fig. 9.1.10)
- if $\alpha \in]0,1[$ then $\lim_{x\to+\infty} x^\alpha = +\infty$ (see fig. 9.1.11)
- if $\alpha \in]-\infty,0[$ then $\lim_{x\to+\infty} x^\alpha = 0$, *i.e.* the $x-$axis is horizontal