CHAPTER 6

DIFFERENTIATION [◊]

6.1 Differentiation of real functions of one real variable

6.1.1 The derivative of a real function

Definition 6.1.1 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. Let $x_0 \in X$ be an accumulation point of X. Let g be the function, also called *incremental quotient at* x_0

$$x \in X - \{x_0\} \quad \to \quad \frac{f(x) - f(x_0)}{x - x_0}. \tag{6.1.1}$$

Evidently x_0 is an accumulation point of $X - \{x_0\}$ too. If there exists the limit at x_0 of the incremental quotient at x_0 and if such limit is a real number, we call such limit the *derivative of the function* f *at the point* x_0 , and denote it by $f'(x_0)$. So,

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} . \tag{6.1.2}$$

We can associate with the function f a function f' whose domain is the set of points x_0 at which the limit (6.1.2) exists. Such f' is called the *derivative*

◊ A. Maceri, *Differentiation*, e-ISBN 978-88-85929-69-2, © Accademica 2020

(or *first derivative*) of f. ◊

Remark 6.1.1 If f' is defined at a point x_0, we say that f is *differentiable at* x_0 . If f' is defined at every point of a set $X' \subseteq X$, we say that f is *differentiable on* X'. ◊

Theorem 6.1.1 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $x_0 \in X$ *be an accumulation point of* X.

The following statements are equivalent:

(6.1.3) $$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

(6.1.4) $$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \quad .$$

Proof. Obvious. ◊

Remark 6.1.2 The derivative (6.1.2) has a *geometric interpretation,* very important for the applications.
Let *f* be a function that maps $X \subseteq \mathbb{R}$ into $\mathbb{R}$. Let O, x, y be, in a plane, a system of orthogonal *Cartesian* axes. Consider, for each $x \in X$, the point P of the plane having abscissa x and ordinate $y = f(x)$. As P varies in X, the point $P = (x, f(x))$ describes in the plane a set of points called *Cartesian diagram of the function* f.
Let (fig. 6.1.1)

- x_0 be a point of X at which f is differentiable
- x any point of X
- $s(x)$ the straight line passing through $(x_0, f(x_0))$ and *secant* at point $(x, f(x))$ the *Cartesian diagram* of the function f
- the *Cartesian diagram* of the function f have *tangent* t_0 in the point $(x_0, f(x_0))$.

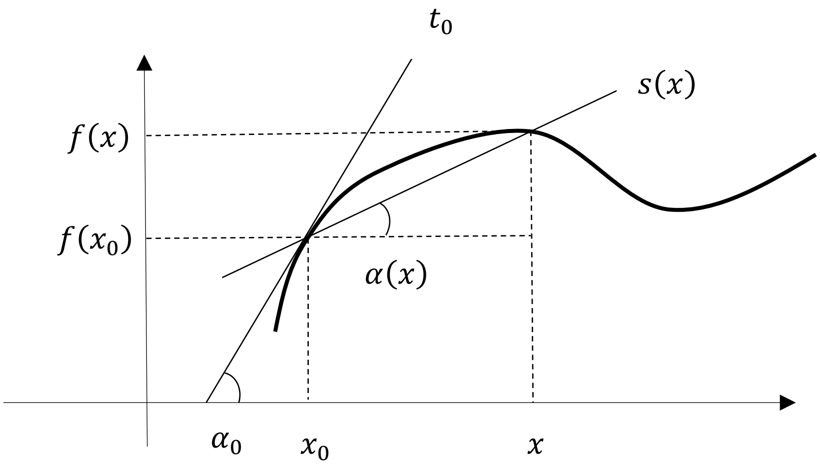

Fig. 6.1.1

Clearly the *angular coefficient* $\operatorname{tg} \alpha(x)$ of the secant straight line is just the incremental quotient at x_0

$$\operatorname{tg} \alpha(x) = \frac{f(x) - f(x_0)}{x - x_0} .$$

Let f be *differentiable at* x_0 . Then, we see in fig. 6.1.1 that when x tends to x_0 , the secant $s(x)$ tends to tangent t_0 . So,

$$\text{tg}\,\alpha_0 = \lim_{x \to x_0} \text{tg}\,\alpha(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = f'(x_0)\,.$$

Therefore, *if you say that* f *is differentiable in* x_0 *and the value of the derivative is* $f'(x_0)$ *, geometrically you say that*

- *the Cartesian diagram of the function* f *at the point* $(x_0, f(x_0))$ *has a tangent* t_0 *whose angular coefficient is* $tg\,\alpha_0 = f'(x_0)$
- when x tends to x_0 , the secant $s(x)$ passing through $(x_0, f(x_0))$ and $(x, f(x))$ tends to tangent t_0 . ◊

Remark 6.1.3 The derivative of any *constant function* is clearly zero. ◊

Theorem 6.1.2 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- f *be differentiable at* $x_0 \in X$.

Then f *is continuous in* x_0 .

Proof. We have

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = \lim_{x \to x_0} \left(\frac{f(x) - f(x_0)}{x - x_0}\right) \cdot (x - x_0)\,.$$

Hence, by theorem 5.1.30

$$\lim_{x \to x_0} \big(f(x) - f(x_0)\big) = f'^{(x_0)} \cdot 0 = 0\,,$$

hence, by theorem 5.1.29

$$\lim_{x \to x_0} f(x) = f(x_0) \ . \ \diamond$$

Remark 6.1.4 The converse of theorem 6.1.2 is not true. In fact, the *absolute value function*

$$x \in \mathbb{R} \quad \to \quad |x| \in [0, +\infty[$$

is continuous in 0, but its incremental quotient at 0 has not limit as $x \to 0$. In fact

$$\frac{|x| - |0|}{x - 0} = \frac{|x|}{x} = \begin{cases} 1 & \text{if } x \in]0, +\infty[\\ -1 & \text{if } x \in]-\infty, 0[\end{cases}$$

and then

$$\lim_{x \to 0^-} \frac{|x| - |0|}{x - 0} = -1 \ , \quad \lim_{x \to 0^+} \frac{|x| - |0|}{x - 0} = 1 \ . \ \diamond$$

On the derivative of the function sum there is the following theorem.

Theorem 6.1.3 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $g: X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$.

Then $f + g$ *is differentiable at* x_0 *and it results*

$$(f + g)'(x_0) = f'(x_0) + g'(x_0) \ . \qquad (6.1.5)$$

Proof. By theorem 5.1.29 we have

$$(f+g)'(x_0) = \lim_{x\to x_0} \frac{(f+g)(x)-(f+g)(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0}\left(\frac{f(x)-f(x_0)}{x-x_0}+\frac{g(x)-g(x_0)}{x-x_0}\right) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$$

$$+\lim_{x\to x_0}\frac{g(x)-g(x_0)}{x-x_0} = f'(x_0)+g'(x_0)\ . \ \diamond$$

On the derivative of the function product there is the following theorem.

Theorem 6.1.4 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $g: X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$.

Then $f \cdot g$ *is differentiable at* x_0 *and it results*

(6.1.6) $$(f\cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)\ .$$

Proof. By theorems 5.1.29, 5.1.30 and by hypothesis of continuity, we have

$$(f\cdot g)'(x_0) = \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x\to x_0}\frac{f(x)g(x)-f(x_0)g(x)+f(x_0)g(x)-f(x_0)g(x_0)}{x-x_0}$$

$$= \lim_{x \to x_0} \left[\left(\frac{f(x) - f(x_0)}{x - x_0} \right) g(x) + f(x_0) \left(\frac{g(x) - g(x_0)}{x - x_0} \right) \right]$$

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0) \ . \ \diamond$$

Remark 6.1.5 Let

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $g: X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$
- $g(x_0) \neq 0$.

In such hypotheses, obviously there exists a neighborhood of x_0 where $\frac{f}{g}$ is defined and x_0 is accumulation point of the definition set of $\frac{f}{g}$. ◊

On the derivative of the function quotient there is the following theorem.

Theorem 6.1.5 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $g: X \subseteq \mathbb{R} \to \mathbb{R}$
- f *and* g *be differentiable at* $x_0 \in X$
- $g(x_0) \neq 0$.

Then $\frac{f}{g}$ *is differentiable at* x_0 *and it results*

$$\text{(6.1.7)} \qquad \left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

Proof. By theorems 6.1.2, 5.1.29, 5.1.30, 5.1.33 and by hypotheses, we have

$$\left(\frac{f}{g}\right)'(x_0) = \lim_{x\to x_0} \frac{\frac{f}{g}(x) - \frac{f}{g}(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{\frac{f(x)}{g(x)} - \frac{f(x_0)}{g(x_0)}}{x - x_0}$$

$$= \lim_{x\to x_0}\left[\frac{1}{g(x)g(x_0)}\left(\frac{f(x)g(x_0) - f(x_0)g(x_0) + f(x_0)g(x_0) - f(x_0)g(x)}{x - x_0}\right)\right]$$

$$= \lim_{x\to x_0}\left\{\frac{1}{g(x)g(x_0)}\left[\left(\frac{f(x) - f(x_0)}{x - x_0}\right)g(x_0) - f(x)\left(\frac{g(x) - g(x_0)}{x - x_0}\right)\right]\right\}$$

$$= \frac{1}{g^2(x_0)}\left(f'(x_0)g(x_0) - f(x_0)g'(x_0)\right) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}.$$

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Remark 6.1.6 Let f be the *identical function*

$$f \;:\; x \in \mathbb{R} \to f(x) = x\,.$$

Then $\forall x \in \mathbb{R}$ it results $f'(x) = 1$. Moreover, repeated application of (6.1.6) shows that the function

$$f \;:\; x \in \mathbb{R} \to f(x) = x^n\,,$$

where $n \in \mathbb{N}$, is differentiable on $\mathbb{R}$ and $\forall x \in \mathbb{R}$ it results $f'(x) = nx^{n-1}$. ◊

On the derivative of the composite function there is the following theorem.

Theorem 6.1.6 *Let*

- $f: X \subseteq \mathbb{R} \to \mathbb{R}$
- $g: f(X) \to \mathbb{R}$
- $x_0 \in X$
- x_0 *be accumulation point for* X
- $f(x_0)$ *be accumulation point for* $f(X)$
- f *be differentiable at* x_0
- g *be differentiable at* $f(x_0)$.

Then the composite function $g \circ f$ *is differentiable at* x_0 *and it results*

(6.1.8) $$(g \circ f)'(x_0) = g'\big(f(x_0)\big)f'(x_0)\,.$$

Proof. To obtain the (6.1.8), we have to prove that

(6.1.9) $$\forall \varepsilon > 0 \quad \exists H_{x_0} : \quad \forall x \in X \cap H_{x_0} - \{x_0\}$$
$$\left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} - g'\big(f(x_0)\big)f'(x_0)\right| < \varepsilon\,.$$

By hypothesis, g is differentiable in $f(x_0)$, and then

(6.1.10) $$\forall \varepsilon > 0 \quad \exists \delta_1 > 0 : \quad \forall y \in f(X)$$
$$(0 < |y - f(x_0)| < \delta_1) \Rightarrow \left(\left|\frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} - g'\big(f(x_0)\big)\right| < \varepsilon\right).$$

By hypothesis, f is differentiable in x_0 , and then

$$(6.1.11) \qquad \forall \rho > 0 \quad \exists W_{x_0} : \quad \forall x \in X \cap W_{x_0} - \{x_0\} \quad \left| \frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \right| < \rho \,.$$

By theorem 6.1.2, f is continuous in x_0 , and then

$$(6.1.12) \qquad \forall \beta > 0 \quad \exists J_{x_0} : \quad \forall x \in X \cap J_{x_0} - \{x_0\} \quad |f(x) - f(x_0)| < \beta \,.$$

Possible cases are $f'(x_0) \neq 0$ and $f'(x_0) = 0$.
Suppose $f'(x_0) \neq 0$. By theorem 5.1.11

$$\exists I_{x_0} : \quad \forall x \in X \cap I_{x_0} - \{x_0\} \qquad \frac{f(x) - f(x_0)}{x - x_0} \neq 0 \,,$$

hence

$$\exists I_{x_0} : \quad \forall x \in X \cap I_{x_0} - \{x_0\} \qquad f(x) - f(x_0) \neq 0 \,,$$

hence

$$(6.1.13) \qquad \forall x \in X \cap I_{x_0} - \{x_0\} \quad \left| \frac{g(f(x)) - g(f(x_0))}{x - x_0} - g'(f(x_0)) f'(x_0) \right|$$

$$= \left| \left(\frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \right) \left(\frac{f(x) - f(x_0)}{x - x_0} \right) - g'(f(x_0)) f'(x_0) \right|$$

$$= \left| \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} \frac{f(x) - f(x_0)}{x - x_0} - \frac{g(f(x)) - g(f(x_0))}{f(x) - f(x_0)} f'(x_0) \right.$$

$$+\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}f'(x_0)-g'(f(x_0))f'(x_0)\Bigg|$$

$$\leq\left|\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}\right|\cdot\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|$$
$$+|f'(x_0)|\cdot\left|\frac{g(f(x))-g(f(x_0))}{f(x)-f(x_0)}-g'(f(x_0))\right|.$$

Moreover, in correspondence of the positive real number 1, by (6.1.10) we have

(6.1.14) $\exists\delta_2>0\ :\ \forall y\in f(X)$ if $0<|y-f(x_0)|<\delta_2$ it results
$$\left|\frac{g(y)-g(f(x_0))}{y-f(x_0)}\right|\leq\left|\frac{g(y)-g(f(x_0))}{y-f(x_0)}-g'(f(x_0))\right|$$
$$+|g'(f(x_0))|<1+|g'(f(x_0))|.$$

To gain the (6.1.9), let us consider now any $\varepsilon\in\,]0,+\infty[$. We notice that, in correspondence of the positive real number

$$\frac{\varepsilon}{2(1+|g'(f(x_0))|)}\ ,$$

by (6.1.11) we have

(6.1.15) $\exists W_{x_0}\ :\ \forall x\in X\cap W_{x_0}-\{x_0\}$ $\quad\left|\frac{f(x)-f(x_0)}{x-x_0}-f'(x_0)\right|$
$$<\frac{\varepsilon}{2(1+|g'(f(x_0))|)}.$$

Moreover, in correspondence of the positive real number

$$\frac{\varepsilon}{2|f'(x_0)|}\ ,$$

by (6.1.10) we have

(6.1.16) $\quad \exists \delta_1 > 0\ :\ \forall y \in f(X)$ such that $(0 < |y - f(x_0)| < \delta_1)$

$$\Rightarrow \left(\left|\frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} - g'\big(f(x_0)\big)\right| < \frac{\varepsilon}{2|f'(x_0)|}\right).$$

Moreover, in correspondence of the positive real number $\delta = \min\{\delta_1, \delta_2\}$, by (6.1.12) we have

(6.1.17) $\quad \exists J_{x_0}\ :\quad \forall x \in X \cap J_{x_0} - \{x_0\} \qquad |f(x) - f(x_0)| < \delta\,.$

Evidently, $H_{x_0} = I_{x_0} \cap W_{x_0} \cap J_{x_0}$ is a neighborhood of x_0 and for every $x \in X \cap H_{x_0} - \{x_0\}$, taking into account the (6.1.13), (6.1.17), (6.1.14), (6.1.15), (6.1.16), we have

$$\left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} - g'\big(f(x_0)\big)f'(x_0)\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

So, if $f'(x_0) \neq 0$ the (6.1.8) is true.
Suppose now $f'(x_0) = 0$. Obviously the (6.1.9) turns in

(6.1.18) $\qquad \forall \varepsilon > 0 \qquad \exists H_{x_0}\ :\qquad \forall x \in X \cap H_{x_0} - \{x_0\}$

$$\left|\frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0}\right| < \varepsilon\,.$$

To gain the (6.1.18), let us consider any $\varepsilon \in\ ]0, +\infty[$.

In correspondence of the positive real number 1, by (6.1.10) we have

(6.1.19) $\exists \delta > 0 \ : \ \forall y \in f(X)$ if $0 < |y - f(x_0)| < \delta$ it results

$$\left| \frac{g(y) - g\big(f(x_0)\big)}{y - f(x_0)} \right| < 1 + \left| g'\big(f(x_0)\big) \right| .$$

Moreover, in correspondence of the positive real number δ, by (6.1.12) we have

(6.1.20) $\exists J_{x_0} \ : \ \forall x \in X \cap J_{x_0} - \{x_0\} \qquad |f(x) - f(x_0)| < \delta$.

We notice that, in correspondence of the positive real number

$$\frac{\varepsilon}{1 + \left| g'\big(f(x_0)\big) \right|} ,$$

by (6.1.11) we have

(6.1.21) $\exists W_{x_0} \ : \ \forall x \in X \cap W_{x_0} - \{x_0\}$

$$\left| \frac{f(x) - f(x_0)}{x - x_0} \right| < \frac{\varepsilon}{1 + \left| g'\big(f(x_0)\big) \right|} .$$

Evidently, $H_{x_0} = W_{x_0} \cap J_{x_0}$ is a neighborhood of x_0 . Let $x \in X \cap H_{x_0} - \{x_0\}$. If $f(x) = f(x_0)$, obviously the (6.1.18) is true. If $f(x) \neq f(x_0)$, taking into account the (6.1.20), (6.1.19), (6.1.21), we have

$$\left| \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{x - x_0} \right| = \left| \frac{g\big(f(x)\big) - g\big(f(x_0)\big)}{f(x) - f(x_0)} \right| \cdot \left| \frac{f(x) - f(x_0)}{x - x_0} \right|$$

$$< \left(1+\left|g'\left(f(x_0)\right)\right|\right)\frac{\varepsilon}{1+\left|g'\left(f(x_0)\right)\right|} = \varepsilon$$

and then the (6.1.18) is true. ◊

On the derivative of the inverse function there is the following theorem.

Theorem 6.1.7 *Let*

- $a, b \in \mathbb{R}\,,\ \ a < b$
- $f : [a, b] \to \mathbb{R}$
- f *be continuous and strictly monotone*
- $x_0 \in [a, b]$
- f *be differentiable at* x_0
- $f'(x_0) \neq 0.$

Then the inverse function f^{-1} *is differentiable at* $f(x_0)$ *and it results*

(6.1.22) $$(f^{-1})'\left(f(x_0)\right) = \frac{1}{f'(x_0)}\,.$$

Proof. To gain the (6.1.22), we have to prove that

$$\lim_{y \to f(x_0)} \frac{f^{-1}(y) - f^{-1}\left(f(x_0)\right)}{y - f(x_0)} = \frac{1}{f'(x_0)}$$

i.e., that

(6.1.23) $\forall \varepsilon > 0 \quad \exists\, \delta > 0 \quad : \quad \forall y \in f(X)$

$$(0 < |y - f(x_0)| < \delta) \Rightarrow \left(\left|\frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} - \frac{1}{f'(x_0)}\right| < \varepsilon\right).$$

Preliminarily we notice that, by hypothesis, $f(x_0)$ is an accumulation point for $f(X)$. In fact, suppose $I_{f(x_0)}$ be any neighborhood of $f(x_0)$. Because f is continuous in x_0, $\exists\, J_{x_0}$ such that $\forall x \in J_{x_0} \cap X - \{x_0\}$ it results $f(x_0) \in I_{f(x_0)}$. Since x_0 is an accumulation point for X, $\exists\, x_1 \in J_{x_0} \cap X - \{x_0\}$ and then $f(x_1) \in I_{f(x_0)}$. Since f is strictly monotonic, we have $f(x_1) \neq f(x_0)$ and then $I_{f(x_0)} \cap f(X) - \{f(x_0)\} \neq \emptyset$.

Let us notice now that the function incremental quotient of f at x_0 converges toward a nonzero limit. As a consequence

$$\lim_{x \to x_0} \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} = \frac{1}{f'(x_0)}$$

i.e.

(6.1.24) $\forall \varepsilon > 0 \quad \exists\, \delta_1 > 0 \quad : \quad \forall x \in X$

$$(0 < |x - x_0| < \delta_1) \Rightarrow \left(\left|\frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} - \frac{1}{f'(x_0)}\right| < \varepsilon\right).$$

Moreover f^{-1} is continuous, since it is the inverse function of a continuous function. Hence

$$\lim_{y \to f(x_0)} f^{-1}(y) = f^{-1}(f(x_0)) = x_0$$

i.e.

(6.1.25) $\forall \beta > 0 \quad \exists\, \delta > 0 \quad : \quad \forall y \in f(X)$

$$(|y - f(x_0)| < \delta) \Rightarrow (|f^{-1}(y) - x_0| < \beta).$$

Now we can build the (6.1.23). Let ε be any positive real number. The (6.1.24) gives us a positive real number δ_1 such that $\forall x \in X$

$$(6.1.26) \qquad (0 < |x - x_0| < \delta_1) \Rightarrow \left(\left| \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} - \frac{1}{f'(x_0)} \right| < \varepsilon \right).$$

In correspondence of the positive real number δ_1 , the $(6.1.25)$ gives us a positive real number δ such that

$$(6.1.27) \qquad \forall y \in f(X) \qquad (|y - f(x_0)| < \delta) \Rightarrow (|f^{-1}(y) - x_0| < \delta_1).$$

Let us consider now any $\forall y \in f(X)$ such that $0 < |y - f(x_0)| < \delta$. Putting $x = f^{-1}(y)$, from the $(6.1.27)$ we obtain $|x - x_0| < \delta_1$. Furthermore $x \neq x_0$. In fact, by absurd suppose $x = x_0$. Hence $y = f(x) = f(x_0)$. Absurd, since $0 < |y - f(x_0)|$.
So, $0 < |x - x_0| < \delta_1$ and then from $(6.1.26)$ we have

$$\left| \frac{1}{\left(\frac{f(x) - f(x_0)}{x - x_0}\right)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

from which

$$\left| \frac{x - x_0}{f(x) - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon$$

from which

$$\left| \frac{f^{-1}(y) - f^{-1}(f(x_0))}{y - f(x_0)} - \frac{1}{f'(x_0)} \right| < \varepsilon . \ \diamond$$

We define now the derivatives and the differentials of higher order.

Definition 6.1.2 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ and let $f': X' \subseteq X \to \mathbb{R}$ be the derivative of f. If there exists the derivative $(f')'$ of f', we denote it f'' and call it the *second derivative* of f. So

(6.1.28) $$f'' = (f')' \; : \; X'' \subseteq X' \to \mathbb{R} . \; \diamond$$

Definition 6.1.3 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$. By induction, the nth *derivative* $f^{(n)}$ (or the *derivative of order n*) of f is defined as the first derivative of the derivative $f^{(n-1)}$ of the $(n-1)$th order. So

(6.1.29) $$f^{(n)} = \left(f^{(n-1)}\right)' \; : \; X^{(n)} \subseteq X^{(n-1)} \to \mathbb{R} . \; \diamond$$

Remark 6.1.7 Of course, the nth derivative of a function f at a point or at subset of X may or may not exist. ⋄

Definition 6.1.4 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$ be a function differentiable at $x \in X$. We call *differential* df (or *first differential*) *of* f at x the first degree polynomial

(6.1.30) $$df \; : \; dx \in \mathbb{R} \to df(dx) = f'(x) \cdot dx . \; \diamond$$

Remark 6.1.8 In the (6.1.30), dx is often called the *differential of* x. ⋄

Definition 6.1.5 Let $f: X \subseteq \mathbb{R} \to \mathbb{R}$, $x \in X$. We call *increment* Δf *of* f at x the function

(6.1.31) $$\Delta f \ : \quad dx \in \mathbb{R} \to \ \Delta f(dx) = f(x+dx) - \ f(x)\ .$$ ◊

Remark 6.1.9 We observe that Δf and df are both infinitesimal in 0. ◊

Theorem 6.1.8 *Let*

- $f : X \subseteq \mathbb{R} \to \mathbb{R}$
- f *differentiable at* $x \in X$.

Then $\Delta f - df$ *is an infinitesimal of higher order than* 1*, i.e.* [6.1.1]

(6.1.32) $$\lim_{dx \to 0} \frac{\Delta f(dx) - df(dx)}{dx} = 0\ .$$

Proof. In fact, it results

$$\lim_{dx \to 0} \frac{\Delta f(dx) - df(dx)}{dx} = \lim_{dx \to 0} \left(\frac{f(x+dx) - f(x)}{dx} - f'(x) \right) = 0\ .$$ ◊

Remark 6.1.10 Theorem 6.1.8 allows us to approximate, in a convenable neighborhood of $x \in X$, any function f differentiable at x. To see this

- we consider any function $f : X \subseteq \mathbb{R} \to \mathbb{R}$ differentiable at $x \in X$
- we consider a neighborhood I_x of x
- we *linearize* (in I_x) the function f, i.e. we approximate (in I_x) f by replacing f with the tangent t at point $\big(x, f(x)\big)$ of the *Cartesian*

[6.1.1] See definition 5.1.46.