CHAPTER 1

SET THEORY♦

1.1 Sets

1.1.1 Basics of Sets

Definition 1.1.1 A *set* is a collection of objects called the *members* (or *elements* or *points*) of the set. If X is a set and x is an element of X, we write

$$x \in X.$$

If X is a set and x is not an element of X, we write

$$x \notin X. \ \diamond$$

Sometimes it is possible to specify a set by listing its members between curly brackets. For example, $\{1,2,\dots,n,\dots\}$ is the set of all positive integers, $1 \in \{1,2\}$, $3 \notin \{1,2\}$.

Remark 1.1.1 Notice that $\{a,b,c\} = \{c,a,b\}$. $\diamond$

Using the *elementary logic*, we say that, if x is an object and X is a set,

♦ A. Maceri, *Set Theory*, e-ISBN 978-88-85929-59-3, © Accademica 2019

then one of the two properties

$$x \in X$$
$$x \notin X$$

is true and the other is false.

Definition 1.1.2 Let I be a set. If to each $i \in I$ there is assigned a set A_i, then the set $\{A_i : i \in I\}$ is called an *indexed family of sets*. In this case, I is called the *index set* for the family and the elements of I are called *indices*. ◊

Definition 1.1.3 Let P and Q be any two property. We say that P *implies* Q and we write

$$P \Rightarrow Q$$

if Q is true every time P is true. We say that P and Q are *equivalent* and we write

$$P \Leftrightarrow Q$$

if it is simultaneously

$$P \Rightarrow Q$$
$$Q \Rightarrow P. \ \diamond$$

If x and y are the same object, we say that x and y are *equal* and we write

$$x = y.$$

If x and y are distinct objects, we say that x and y are *distinct* and we write

$$x \neq y.$$

Definition 1.1.4 Let A and B be any two sets. If each member of A

if also member of *B,* we say that *A* is a *subset* (or a *part*) of *B* (or that *A* is *contained* in *B*) or that *B contains A* and we write

$$A \subseteq B$$

or

$$B \supseteq A. \diamond$$

Definition 1.1.5 Let *A* and *B* be any two sets. If *A* and *B* have precisely the same members, we say that *A* is *equal* to *B* and we write

$$A = B. \diamond$$

Obviously

$$(A = B) \Leftrightarrow (A \subseteq B \text{ and } B \subseteq A).$$

Definition 1.1.6 If $A \subseteq B$ and $A \neq B$ we say that *A* is a *proper subset* of *B* and we write $A \subset B$ or $B \supset A$. ◊

Definition 1.1.7 The set containing no elements at all is called the *void set* (or *empty set*) and is denoted by the symbol $\emptyset$. ◊

The set $\emptyset$ is clearly a subset of every set.

1.1.2 Operations on Sets

Definition 1.1.8 Let *A* and *B* be given sets. We call *union* of *A* and *B*, and denote by $A \cup B$, the set consisting of all elements which belong to at least one of the sets *A* and *B*. In symbols

$$A \cup B = \{x : x \in A \text{ or } x \in B\}. \diamond$$

For example, we have $\{1,2\} \cup \{2,3\} = \{1,2,3\}$.

Definition 1.1.9 We call *union* of the indexed family of sets $\{A_i : i \in I\}$, and denote by $\cup_{i \in I} A_i$, the set consisting of all elements which belong to at least one of the sets A_i. ◊

Definition 1.1.10 Let *A* and *B* be given sets. We call *intersection* of *A* and *B*, and denote by $A \cap B$, the set consisting of all elements which belong to both *A* and *B*. In symbols

$$A \cap B = \{x : x \in A \text{ and } x \in B\}. \diamond$$

For example, we have $\{1,2\} \cup \{1,2,3\} = \{1,2\}$.

Definition 1.1.11 We call *intersection* of the indexed family of sets $\{A_i : i \in I\}$, and denote by $\cap_{i \in I} A_i$, the set consisting of all elements which belong to every one of the sets A_i. ◊

From the above definitions it immediately follows that the operations $\cup$ and $\cap$ are *commutative*, i.e., that

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A,$$

associative, i.e., that

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

and obey the following *distributive laws*

$$(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$$

$$(A \cap B) \cup C = (A \cup C) \cap (B \cup C).$$

Definition 1.1.12 Let *A* and *B* be given sets. We say that *A* and *B* are *disjoint* if they have no elements in common, i.e., if

$$A \cap B = \emptyset. \ \diamond$$

Definition 1.1.13 Let $\mathcal{F}$ be a family of sets such that $A \cap B = \emptyset$ for every pair of sets A, B in $\mathcal{F}$. Then the sets in $\mathcal{F}$ are said to be *pairwise disjoint.* ◊

Definition 1.1.14 Let X be given set, A a subset of X. We call *complement* of *A* and denote by A^c (or by $X - A$) the set of all elements of *X* which do not belong to *A*. ◊

Remark 1.1.2 Let *X* be a given set, $\{A_i: i \in I\}$ an indexed family of subset of *X*. We easily verify that $\cup_{i \in I} A_i$ and $\cap_{i \in I} A_i$ are both subset of *X*. ◊

Remark 1.1.3 Let A, B, C be any sets. We easily verify that

$$A \cup A = A$$
$$A \cap A = A$$
$$A \cup \emptyset = A$$
$$A \cap \emptyset = \emptyset$$
$$A \subset A \cup B$$
$$A \cap B \subset A. \ \diamond$$

Theorem 1.1.1 [De Morgan's Laws] *Let X be given set,* $\{A_i: i \in I\}$ *an indexed family of subset of X. It results*

(1.1.1) $$(\cup_{i \in I} A_i)^c = \cap_{i \in I} (A_i)^c$$
(1.1.2) $$(\cap_{i \in I} A_i)^c = \cup_{i \in I} (A_i)^c.$$

Proof. To prove (1.1.1), suppose $x \in (\cup_{i\in I} A_i)^c$, i.e., x does not belong to any of the sets A_i. It follows that x belongs to each of the complements $(A_i)^c$ and hence $x \in\cap_{i\in I} (A_i)^c$. Thus $(\cup_{i\in I} A_i)^c \subseteq \cap_{i\in I} (A_i)^c$.
Conversely, suppose $x \in\cap_{i\in I} (A_i)^c$, so that x belongs to every set $(A_i)^c$, i.e., x does not belong to any of the sets A_i. Hence x does not belong to the union $\cup_{i\in I} A_i$, and then $x \in (\cup_{i\in I} A_i)^c$. Thus $\cap_{i\in I} (A_i)^c \subseteq (\cup_{i\in I} A_i)^c$.
This proves (1.1.1).
The (1.1.2) can be proved similarly. ⋄

Definition 1.1.15 Let *B* be a given set, *A* be a subset of *B*. We call *cover* of *A* any indexed family $\{A_i: i \in I\}$ of subset of *B* such that

$$A \subseteq \cup_{i\in I} A_i \ . \ \diamond$$

Definition 1.1.16 Let *X* be a given set. Any family $\{X_i: i \in I\}$ of pairwise disjoint subset of *X* such that

$$\cup_{i\in I} X_i = X$$

is called a *partition* (or *decomposition*) of *X*. ⋄

Definition 1.1.17 Let *X* and *Y* be given sets. We call *Cartesian*[1.1.1] *product of X and Y* the set [1.1.2]

$$X \times Y = \{(x, y): x \in X, y \in Y\}.$$

Every element (x, y) of $X \times Y$ is called *ordered pair*, where *x* is called the *first coordinate* of (x, y) and *y* is called the *second coordinate* of (x, y).⋄

Remark 1.1.4 If (x, y) and (a, b) are two ordered pairs, we write

---

[1.1.1] To honor René Descartes, La Haye (French) 1506 -.Stockholm 1650.
[1.1.2] The symbol " : " means "such that".

$(x, y) = (a, b)$ if and only if $x = a$ and $y = b$. Thus $(1,5) \neq (5,1)$ while $\{1,5\} = \{5,1\}$. ⋄

Definition 1.1.18 Let *n* be a positive integer number and $X_1, \dots, X_n$ be *n* sets. We call *Cartesian product of* $X_1, \dots, X_n$ the set

$$X_1 \times \dots \times X_n = \{x = (x_1, \dots, x_n): x_i \in X_i \ \ \forall i \in \{1, \dots, n\}\}.$$

We call, $\forall i \in \{1, \dots, n\}$, the point $x_i \in X_i$ the *i*th *coordinate* of the *ordered n tuple* $(x_1, \dots, x_n) \in X_1 \times \dots \times X_n$. ⋄

1.1.3 Relations

Definition 1.1.19 Let *X* be a set. Any subset *R* of $X \times X$ is called a (*binary*) *relation on X*. If $(x, y) \in R$ we say that R is verified by the ordered pair (x, y) and we write

$$x\, R\, y\,. \ \diamond$$

Definition 1.1.20 Let *X* be a set, R be a relation on *X*. We call *domain* of R the set [1.1.3]

$$\operatorname{dom} R = \{x \in X : \exists (x, y) \in R\}.$$

We call *range* of R the set

$$\operatorname{rng} R = \{y \in X : \exists (x, y) \in R\}. \ \diamond$$

[1.1.3] The symbol "∃" means “it exists”.

Definition 1.1.21 Let R be a relation on *X*. We say that R is *reflexive* if $\forall x \in \operatorname{dom} R$

$$x \, R \, x, \tag{1.1.3}$$

symmetric if

$$x \, R \, y \;\Rightarrow\; y \, R \, x, \tag{1.1.4}$$

antisymmetric if

$$(x \, R \, y \;\text{ and }\; y \, R \, x\,) \Rightarrow\; (x = y), \tag{1.1.5}$$

transitive if

$$(x \, R \, y \;\text{ and }\; y \, R \, z) \;\Rightarrow\; (x \, R \, z). \;\diamond \tag{1.1.6}$$

Definition 1.1.22 Let R be a relation on *X* such that $\operatorname{dom} R = X$. We say that R is an *equivalence relation on X* if it is reflexive, symmetric and transitive. ◊

If R is an equivalence relation on *X*, the element $x \, R \, y$ is often denoted $x \equiv y$ and we say that *x* is *equivalent* to *y* by R.

Definition 1.1.23 Let R be an equivalence relation on *X* and $x \in X$. The set

$$R_x = \{y \in X : y \equiv x\}$$

is called *equivalence class of X containing x*. ◊

We also say that R_x is *represented* by any one of its elements. Besides, if $y \in R_x, y$ is said a *representative* of R_x. It is easy to check that the family

(1.1.7) $$\frac{X}{R_x} = \{R_x : x \in X\}$$

of all such equivalence classes is a family of nonvoid pairwise disjoint sets and its union is *X* and then is a partition of *X*.

Definition 1.1.24 The partition (1.1.7) of *X* is called *quotient set of X*. ◊

Definition 1.1.25 Let R be a relation on *X* such that $dom\, R = X$. We say that R is a *partial order on X* if it is reflexive, antisymmetric and transitive. If R is a partial order on *X,* we usually write $a \leq b$ or $b \geq a$ instead $a\, R\, b$. ◊

Definition 1.1.26 The notation $a < b$ (or $b > a$) indicates that $a \leq b$ and $a \neq b$. ◊

Definition 1.1.27 If *X* is a set provided with a partial order, we say that *X* is a *partially ordered set*. ◊

Definition 1.1.28 We say that X is an *ordered set* (or a *totally ordered set*) if

(1.1.8) X is a partially ordered set,
(1.1.9) $\forall x, y, z \in X$ one and only one of the statements

$$x < y, \qquad x = y, \qquad y < x$$

is true (*trichotomy property*). ◊

Definition 1.1.29 Let *X* be an ordered set, and $Y \subseteq X$. If there exists a $\beta \in X$ such that $x \leq y$ for every $x \in Y$, we say that Y is *bounded above*, and call β an *upper bound* for Y. ⋄

Definition 1.1.30 Let X be an ordered set, and $Y \subseteq X$. If there exists an $\alpha \in X$ such that $\alpha \leq x$ for every $x \in Y$, we say that Y is *bounded below*, and call α a *lower bound* for Y. ⋄

Definition 1.1.31 Let X be an ordered set, and $Y \subseteq X$. If Y has both an upper bound and a lower bound, then we say that Y is *bounded*.⋄

Definition 1.1.32 Let X be an ordered set, and $Y \subseteq X$. By a *maximum* of Y we mean an element of Y, denoted $\max Y$, such that $\max Y$ is an upper bound for Y. ⋄

Remark 1.1.5 Let X be an ordered set, and $Y \subseteq X$. It is clear that Y can have at most one maximum. ⋄

Definition 1.1.33 Let X be an ordered set, and $Y \subseteq X$. By a *minimum* of Y we mean an element of Y, denoted $\min Y$, such that $\min Y$ is a lower bound for Y. ⋄

Remark 1.1.6 Let X be an ordered set, and $Y \subseteq X$. It is clear that Y can have at most one minimum. ⋄

Definition 1.1.34 Let X be an ordered set, $Y \subseteq X$, $Y \neq \emptyset$. We say that X has the *least-upper-bound property* if there exists an element of X, called *supremum* (or *least upper bound*) of Y, and denoted

$$\sup Y,$$

such that

(1.1.10) $\sup Y$ is an upper bound for Y,
(1.1.11) if γ is any upper bound for Y, then $\sup Y \leq \gamma$. ⋄

Remark 1.1.7 Let X be an ordered set, and $Y \subseteq X$. We underline that if $\alpha = \sup Y$ exists, then α may or may not be a member of Y. Furthermore, we underline that, if Y has the least-upper-bound property, denoting B the set of the upper bounds of Y, it results

$$\sup Y = \min B. \diamond$$

Remark 1.1.8 Let X be an ordered set, and $Y \subseteq X$. It is clear that Y can have at most one supremum. ⋄

Definition 1.1.35 Let X be an ordered set, $Y \subseteq X$, $Y \neq \emptyset$. We say that X has the *greatest-lower-bound property* if there exists an element of X, called *infimum* (or *greatest lower bound*) of Y, and denoted

$$\inf Y,$$

such that

(1.1.12) $\inf Y$ is a lower bound for Y,
(1.1.13) if δ is a lower bound for Y, then $\inf Y \geq \delta$. ⋄

Remark 1.1.9 Let X be an ordered set, and $Y \subseteq X$. We underline that if $\alpha = \inf Y$ exists, then α may or may not be a member of Y. Furthermore, we underline that, if Y has the greatest-lower-bound property, denoting A the set of the lower bounds of Y, it results

$$\inf Y = \max A. \diamond$$

Remark 1.1.10 Let X be an ordered set, and $Y \subseteq X$. It is clear that Y can have at most one infimum. ⋄

Theorem 1.1.2 *Suppose X is an ordered set with the least-upper-bound property, $Y \subseteq X$, Y is not empty, and Y is bounded below. Let A be the set of all lower bounds of Y. Then*

$$\alpha = \max A$$

exists in X, and $\alpha = \inf Y$.

Proof. Obviously $A \subseteq X$. Moreover, since Y is bounded below, A is not empty. Since $A = \{y \in X : y \leq x \quad \forall x \in Y\}$, every $x \in Y$ is an upper bound of A, hence A is bounded above. By hypothesis, X has the least-upper-bound property. Hence $\exists \alpha \in X: \alpha = \sup A$.
Since $\alpha = \sup A$, we have that α is greater or equal than every upper bound of A. So, if $\mu < \alpha$ then μ is not an upper bound of A, hence $\mu \notin Y$. In fact, we have already seen that every member of Y is an upper bound of A. It follows that, for every $y \in Y$, we have $\alpha \leq y$. Thus $\alpha \in A$.
Hence $\alpha = \max A$, hence $\alpha = \inf Y$. ◊

Remark 1.1.11 Theorem 1.1.2 show that that every ordered set with the least-upper-bound property also has the greatest-lower-bound property.◊

1.1.4 Functions

Definition 1.1.36 Let *X* and *Y* be sets, X' be a subset of *X*. A rule *f* associating a unique $y \in Y$ with each $x \in X'$ is called a (*single-valued*) *function from* X' *into Y*. The set X' is called the *domain (of definition)* of *f* and is denoted by dom *f*. The unique element *y* of *Y* (associated by *f* with the element *x* of X'), is called the *value of f at x* (or the *image of x under f*) and denoted by $f(x)$. We say that *f maps X into Y* and we write

$$f: X \to Y$$

or

$$f: x \in X' \subseteq X \to f(x) \in Y.$$

The set $\{f(x) \in Y: x \in X'\}$ is called the *range* of f (or *image* of X') and is denoted $f(X')$ or rng$f$. ◊

Remark 1.1.12 Obviously $f(X') \subseteq Y$ and in general an element of $f(X')$ is the value of f at several elements of X'. ◊

Definition 1.1.37 If $\text{dom}\, f = X$ and $f(X') = Y$ we say that f is a *function from X onto Y.* ◊

Remark 1.1.13 A function is also called *single-valued relation* or *mapping* or *transformation* or *operation* or *correspondence* or *application.*◊

Definition 1.1.38 Let f be a function that maps $X' \subseteq X$ into *Y*. If

$$\forall x, z \in X' \qquad x \neq z \Rightarrow f(x) \neq f(z),$$

we say that f is a *reversible* function. ◊

Remark 1.1.14 Let $f: X' \subseteq X \to Y$ be any reversible function. Obviously $\forall y \in \text{rng}f$ there exists one and only one $x \in X'$ such that $f(x) = y$. ◊

Definition 1.1.39 Let f be a function that maps $X' \subseteq X$ into Y. If f is a reversible function, the (single-valued) function f^{-1}

$$\forall y \in \text{rng}\, f \to \text{the unique } x \in X' \text{ such that } f(x) = y$$

is called the *inverse* of f. ◊

Remark 1.1.15 If $f: X' \subseteq X \to Y$ is a reversible function, obviously $\text{dom}f^{-1} = \text{rng}f$ and $\text{rng}f^{-1} = \text{dom}f$. ◊

Definition 1.1.40 Let f be a function that maps X onto Y. If f is a reversible function, we say that f is a *one to one* (or *biunique*) *correspondence from X onto Y.* ◊

Thus, to say that f is a one to one (or biunique) correspondence from X onto Y simply means that each element of Y is the correspondent (by f) of one and only one element of X and each element of X is the correspondent (by f^{-1}) of one and only one element of Y.

Definition 1.1.41 If $f\colon X \to Y$ is a function and $A \subset X$, we define the *restriction of* f *to* A to be the function $f_A : A \to Y$ such that

$$x \in A \;\rightarrow\; f_A(x) = f(x) \in Y. \;\diamond$$

Remark 1.1.16 Usually the restriction f_A of f to A is denoted by the same symbol f of the function. ◊

Definition 1.1.42 If $f\colon X \to Y$ is a function and $X \subset B$, we define the *extension of* f *to* B to be the function $f_B : B \to Y$ such that

$$x \in X \;\rightarrow\; f_B(x) = f(x) \in Y. \;\diamond$$

Remark 1.1.17 Usually the extension f_B of f to B is denoted by the same symbol f of the function. ◊

Definition 1.1.43 Let $f\colon X \to Y$ and $g\colon Y \to Z$ be any functions. We define the *composite* function $g \circ f$ to be the function

$$x \in X \;\rightarrow\; g \circ f(x) = g\big(f(x)\big) \in Y. \;\diamond$$

Definition 1.1.44 Let X be any set and A be any subset of X. The function χ_A with domain X and range contained in $\{0,1\}$ such that

$$\chi_A(x) = \begin{cases} 1 & \text{if } \ x \in A \\ 0 & \text{if } \ x \in X - A, \end{cases}$$

is called the *characteristic function of A.* ◊

Remark 1.1.18 The characteristic functions are very useful in the *Mathematical Analysis*. ◊

Definition 1.1.45 A *sequence* is a function having $\mathbb{N}$ [1.1.4] as its domain. If x is a sequence, we will write x_n instead of $x(n)$ for the value of x at n. The value x_n is called the n^{th} *term* of the sequence. The sequence x whose n^{th} term is x_n will be denoted by

$$x_1, \dots, x_n, \dots$$

or simply

$$\{x_n\}.$$

If Y is a set and if $x_n \in Y \quad \forall n \in \mathbb{N}$, then $\{x_n\}$ is said to be a *sequence in* Y, or a *sequence of elements of* Y. ◊

Definition 1.1.46 Two sets A and B are said *equivalent* if there exists some one-to-one function from A onto B. ◊

Definition 1.1.47 A set X is said *finite* if either $X = \emptyset$ or else exists some $n \in \mathbb{N}$ such that X is equivalent to $\{j \in \mathbb{N} : 1 \leq j \leq n\}$.
All sets that are not finite are said to be *infinite*.
A set equivalent to $\mathbb{N}$ is said *denumerable* (or *enumerable*).
A set that is either finite or denumerable is said to be *countable*.
Any set that is not countable is called uncountable. ◊

[1.1.4] We denote by $\mathbb{N}$ the set of all positive integer numbers.