CHAPTER 4

STATICS*

4.1 Statics of structures

4.1.1 Rigid bodies statics

Let us consider an any one-dimensional plane structure, whose axis is a broken line. We have seen in the chapter 2 that to analyze the structure it needs first of all to do the kinematic analysis. It consists in the determination of

- the lability degree $\boldsymbol{r}$, that is the number of the necessary parameters to individualize the position assumed by the bodies of the structure when it happens an any kinematic mechanism allowed by the constraints
- the hyperstatic degree $\boldsymbol{h}$, that is the number of the freedom degrees that the constraints can prevent but they don't prevent. In other terms $\boldsymbol{h}$ individualizes the excess of constraints in comparison to those strictly necessary to confer to the structure the lability degree $\boldsymbol{r}$.

As already underlined in the chapter 2, in the general case to avoid the collapse of the structure the opportune value of $\boldsymbol{r}$ is 0. In fact with such value the present constraint are sufficient to prevent any beam movement. As to $\boldsymbol{h}$, obviously the more superabundant constraints there are the more the structure is sure. But, as we will see, the more superabundant

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constraints there are the more the structural calculus is complex.

Let us suppose that the load applied on the structure doesn't vary in the time. In such hypothesis the structural analysis is called *static analysis* or simply *Statics*.

In classical *Physics* we say that *a solid body* $\mathfrak{C}$ *is in static equilibrium if the forces, distributed and/or concentrated, applied on it they are a vector system equivalent to zero*[4.1.1].

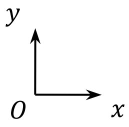

Fig. 4.1.1

Let

- *O,x,y* be a Cartesian orthogonal reference frame (fig. 4.1.1)
- $\boldsymbol{\Sigma}$ be a mechanical load applied on $\mathfrak{C}$, constituted by the $n \in \boldsymbol{N}$ forces $\boldsymbol{F_1}, \dots, \boldsymbol{F_n}$ of $\mathfrak{R}^2$, respectively applied on the straight lines $r_1, \dots, r_n$
- $(\boldsymbol{F_i})_x$ [resp. $(\boldsymbol{F_i})_y$] be the orthogonal component on *x* [resp. *y*] of $\boldsymbol{F_i}$, $\forall i \in \{1, \dots, n\}$
- *P* be any point of $\mathfrak{R}^2$
- $\boldsymbol{M_P}$ be the polar resultant moment as to pole *P* of $\boldsymbol{\Sigma}$.

Because of the [1.3.21], if $\mathfrak{C}$ is in static equilibrium, it must be

$$(4.1.1) \qquad \sum_{i=1}^{n} (\boldsymbol{F_i})_x = 0$$

$$(4.1.2) \qquad \sum_{i=1}^{n} (\boldsymbol{F_i})_y = 0$$

[4.1.1] See section 1.3.5.

(4.1.3) $$\boldsymbol{M_P} = \mathbf{0}\,.$$

The three equations (4.1.1), (4.1.2), (4.1.3) are called the *three cardinal equations* of *Statics*. In particular

- the (4.1.1) is called *equation of equilibrium of* $\mathfrak{C}$ *to horizontal translation*
- the (4.1.2) is called *equation of equilibrium of* $\mathfrak{C}$ *to vertical translation*
- the (4.1.3) is called *equation of equilibrium of* $\mathfrak{C}$ *to rotation around P*.

In the hypothesis of small deformations, the deformed configuration of the body $\mathfrak{C}$ is very near to the initial one. So we can apply to the initial configuration of the body the following postulate of the classical *Physics*, called *principle of dissection*

[4.1.1] *Let us consider the following problem 1, in which*

- *an any plane body* $\mathfrak{C}$ *is submitted to any mechanical and/or thermal load* $\boldsymbol{\Sigma}$
- $\boldsymbol{\Sigma}$ *doesn't vary in the time and is in static equilibrium*
- $\mathfrak{C}$ *occupies the region V of the plane*
- *the boundary S of V is a regular curve*
- *the load is applied on a part* S_p *of S.*

We now consider the following problem 2, in which we

- *remove from the body* $\mathfrak{C}$ *any part and consider the remaining part* $\mathfrak{C}_0$
- *denote with* V_0 *the region of plane occupied by* $\mathfrak{C}_0$
- *suppose that the boundary* S_0 *of* V_0 *is a regular curve of* $\mathfrak{R}^2$
- *apply on* $\mathfrak{C}_0$ *the mechanical and/or thermal load acting on it before we isolated* $\mathfrak{C}_0$
- *apply on* $S_p \cap S_0$ *the mechanical and/or thermal load insisting on it before we isolated* $\mathfrak{C}_0$
- *apply on* $S_0 - S$ *the stress vectors that the removed part of* $\mathfrak{C}$

charged on the isolated one $\mathfrak{C}_0$. Clearly such stress vectors are now a distributed load for $\mathfrak{C}_0$.

Well in such hypotheses

- *in the problem 2 the isolated body $\mathfrak{C}_0$ is in static equilibrium*
- *in the common points the bodies $\mathfrak{C}$ of problem 1 and $\mathfrak{C}_0$ of problem 2 have just the same state of strain and stress*
- *the fields of displacement of the bodies $\mathfrak{C}$ of problem 1 and $\mathfrak{C}_0$ of problem 2 differ at most by a rigid translation and/or by a rigid rotation.* ◊

We immediately observe that the principle of dissection allows us to calculate the constraint reactions. In fact it is enough to apply the dissection principle to any beam $\mathfrak{C}$ of the structure. Precisely we isolate $\mathfrak{C}$ removing from $\mathfrak{C}$ all the constraints. Before the removal, such constraints performed on $\mathfrak{C}$ some forces, called *constraint reactions*. The dissection principle obliges us to replace the constraints with their constraint reactions. Such way the isolated beam $\mathfrak{C}$ is loaded by a system of forces $\boldsymbol{\Sigma}$ constituted by the mechanical load applied on $\mathfrak{C}$ before the removal and by the constraint reactions applied on $\mathfrak{C}$ after the removal. Because of the dissection principle [4.1.1], we obtain that $\boldsymbol{\Sigma} = 0$. So every beam of structure furnishes us three cardinal equations of *Statics* in the unknown constraint reactions. The constraints are been described in detail in chapter 2. In particular it has been specified that, denoting with S the constrained cross-section

Fig. 4.1.2

- the *fixed joint* has kinematic order 3 (fig. 4.1.2). Its reaction on the beam is a system $\boldsymbol{\Sigma}$ of distributed forces. If the resultant $\boldsymbol{R}$ of $\boldsymbol{\Sigma}$ is not null
  - because of the [1.3.18] $\boldsymbol{\Sigma}$ is equivalent to system constituted by the only resultant $\boldsymbol{R}$ applied on the central axis of $\boldsymbol{\Sigma}$

– the constitution of the fixed joint doesn't impose limitations neither to the intensity of $\boldsymbol{R}$, neither to its direction, neither to its verse, neither to the position of the central axis.

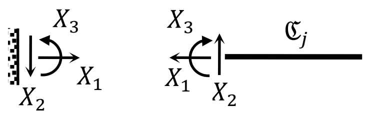

Fig. 4.1.3

If the resultant $\boldsymbol{R}$ of $\boldsymbol{\Sigma}$ is null then, because of the [1.3.23], $\boldsymbol{\Sigma}$ is equivalent to system constituted by the only polar resultant moment of $\boldsymbol{\Sigma}$ as to any point P of $\mathfrak{R}^2$.
If $\boldsymbol{R} \neq \boldsymbol{0}$, denoting

– with G the barycentre of S
– $\boldsymbol{X_1}$ [resp. $\boldsymbol{X_2}$] the horizontal [resp. vertical] component of $\boldsymbol{R}$
– $\boldsymbol{X_3}$ the polar resultant moment of $\boldsymbol{\Sigma}$ as to G,

obviously $\boldsymbol{\Sigma}$ is equivalent to vector system $\{\boldsymbol{X_1}, \boldsymbol{X_2}, \boldsymbol{X_3}\}$ (fig. 4.1.3). As a consequence the reaction of the fixed joint is simulated by the ordered triplet of real numbers (X_1, X_2, X_3).

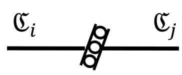

Fig. 4.1.4

○ the *sliding joint* has kinematic order 2 (fig. 4.1.4). Its reaction on the beam is a system $\boldsymbol{\Sigma}$ of parallel concentrated forces orthogonal to double track. If the resultant $\boldsymbol{R}$ of $\boldsymbol{\Sigma}$ is not null

– because of the [1.3.18] $\boldsymbol{\Sigma}$ is equivalent to system constituted by the only resultant $\boldsymbol{R}$ applied on the central axis of $\boldsymbol{\Sigma}$

– the constitution of the fixed joint doesn't impose limitations neither to the intensity of $\boldsymbol{R}$, neither to its verse, neither to the position of the central axis.

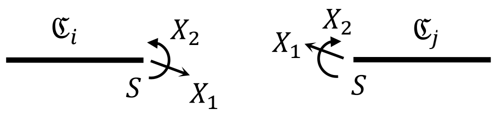

Fig. 4.1.5

If the resultant $\boldsymbol{R}$ of $\boldsymbol{\Sigma}$ is null then, because of the [1.3.23], $\boldsymbol{\Sigma}$ is equivalent to system constituted by the only polar resultant moment of $\boldsymbol{\Sigma}$ as to any point P of $\mathfrak{R}^2$.
If $\boldsymbol{R} \neq \boldsymbol{0}$, denoting

– with G the barycentre of S
– $\boldsymbol{X_1}$ the resultant $\boldsymbol{R}$
– $\boldsymbol{X_2}$ the polar resultant moment of $\boldsymbol{\Sigma}$ as to G,

obviously $\boldsymbol{\Sigma}$ is equivalent to system $\{\boldsymbol{X_1}, \boldsymbol{X_2}\}$ (fig. 4.1.5). As a consequence the reaction of the fixed joint is simulated by the ordered couple of real numbers (X_1, X_2).

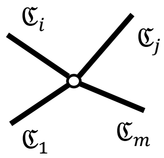

Fig. 4.1.6

- the *hinge* connects, only allowing relative rotations around the center of the hinge, $m \in \boldsymbol{N}$ one-dimensional bodies $\mathfrak{C}_1, \dots, \mathfrak{C}_m$ of the structure (fig. 4.1.6). It has kinematic order $2(m-1)$. As clarified better in the section 2.1.2, the device is such that every beam $\mathfrak{C}_i$ can receive contact pressures from the pin but not from the other hinged beams. The pin can rotate in the hole without attrition and acts on the hole of the beam $\mathfrak{C}_i$ a system $\boldsymbol{\Sigma}$ of distributed radial forces. As proven in problem 1.3.1, $\boldsymbol{\Sigma}$ is equivalent to system constituted by its resultant $\boldsymbol{R}$ applied on a straight line passing through the center of the pin. Denoting with $\boldsymbol{X_1}$ [resp. $\boldsymbol{X_2}$] the horizontal [resp. vertical] component of $\boldsymbol{R}$, obviously $\boldsymbol{\Sigma}$ is equivalent to system $\{\boldsymbol{X_1}, \boldsymbol{X_2}\}$ (fig. 4.1.7).

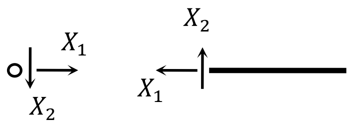

Fig. 4.1.7

Clearly the pin

- introduce 2 static unknowns for every hinged beam
- must be in equilibrium to horizontal and vertical translation.

So the hinge that connects m beams introduce $2(m-1)$ static unknowns.

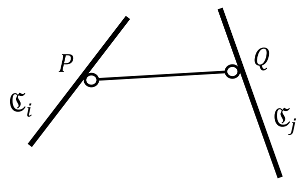

Fig. 4.1.8

- the *pendulum* connects, by two hinges inserted in the holes P and Q, two one-dimensional bodies $\mathfrak{C}_i, \mathfrak{C}_j$ of the structure (fig. 4.1.8). It has kinematic order 1. The device is such that the body PQ is submitted only to radial pressures performed on the holes by the two pins. The equilibrium of PQ imposes that such radial pressures are a null vector system. So the resultant of the left radial pressures, whose straight line of action pass through the center of the left hole, and the resultant of the right radial pressures, whose straight line of action pass through the center of the right hole, must constitute a null vector system. Because of the [1.3.25] these resultants must have the same straight line of action, and then pass through the centers of the holes. Furthermore these resultants, that we denote with $\boldsymbol{X_1}$, must have equal intensity and opposite verse.

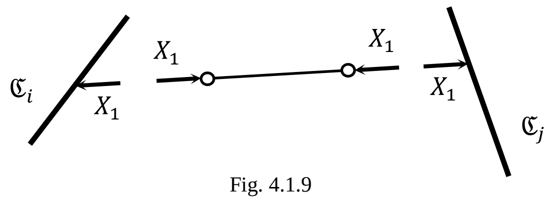

Fig. 4.1.9

Because of the principle of action and reaction of the *Physics*, the bodies $\mathfrak{C}_i$ and $\mathfrak{C}_j$ practice on the pendulum actions $\boldsymbol{X_1}$ having equal line of action, equal intensity but opposite verse (fig. 4.1.9). Conclusively, the pendulum introduce 1 static unknown.

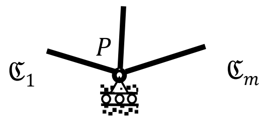

Fig. 4.1.10

- the *bogie* (fig. 4.1.10) is a constraint constituted by a device that
  - it is connected, by one pin inserted in the hole P, with $m \in \boldsymbol{N}$ one-dimensional bodies $\mathfrak{C}_1, \dots, \mathfrak{C}_m$
  - it has a circular pin that can rotate in the holes without attrition, so that it exerts on the holes radial pressures. It is organized by forked pins, so that neither two any hinged bodies have points of contact. Moreover neither the hinged bogie and any hinged body have points of contact. As a consequence every connected body can receive contact pressures from the pin but not from any other

hinged body

– it is connected, by two or more wheels, having equal diameter and centers on any straight line *r*, with a double track (parallel to *r*, having track's distance equal to diameter and integral with a rigid foundation) in which the wheels can roll without attrition (fig. 4.1.11).

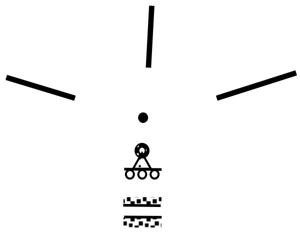

Fig. 4.1.11

The bogie has kinematic order $2m-1$. The pin exerts on any connected body $\mathfrak{C}_i$ a system $\boldsymbol{\Sigma_i}$ of radial pressures that (as proven in problem 1.3.1) is equivalent to the resultant $\boldsymbol{R_i}$ passing through the pin center. A s consequence $\boldsymbol{\Sigma_i}$ is equivalent to the system constituted by the horizontal component $\boldsymbol{X_{i1}}$ of $\boldsymbol{R_i}$ passing through the pin center and by the vertical component $\boldsymbol{X_{i2}}$ of $\boldsymbol{R_i}$ passing through the pin center. So, because of the action and reaction principle, the bodies $\mathfrak{C}_1, \dots, \mathfrak{C}_m$ exert on the pin $2m$ unknowns forces $X_1, \dots, X_{2m}$. On the other hand, the double track (integral with the rigid foundation) exert on the bogie wheels (that have to roll without attrition) some concentrated forces orthogonal to double track. So, if we isolate the bogie, disconnecting the pin from the m bodies and the double track from the wheels, clearly the isolated bogie is submitted to

– actions $X_1, \dots, X_{2m}$ made, by the action and reaction principle, from the m bodies
– some actions normal to double track made on the wheels from the double track.

The equilibrium of the bogie to translation according the direction of the double track furnishes one equation where only appear the unknowns $X_1, \ldots, X_{2m}$. This way we conclude that the bogie that connects $m \in \boldsymbol{N}$ bodies of the structure is simulated by $2m-1$ static unknowns.

- the *double sliding joint* is a constraint constituted by a device $\mathfrak{C}$ that connects two one-dimensional bodies $\mathfrak{C}_i, \mathfrak{C}_j$ of the structure (fig. 4.1.12)
  - allowing their relative translation according any direction
  - not allowing their relative rotation.

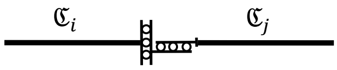

Fig. 4.1.12

This constraint can be made installing on the one-dimensional body $\mathfrak{C}_i$ a double track, having direction r and track's distance *d* and another double track, having direction s orthogonal to r and track's distance *d* (fig. 4.1.12). On a rigid element $\mathfrak{C}$ they are

- two or more wheels all having diameter *d*, whose centers are on a straight line parallel to *r*, that all can roll without attrition in the double tracks
- two or more wheels all having diameter *d*, whose centers are on a straight line parallel to *s*, that all can roll without attrition in the double tracks.

The double sliding joint has kinematic order 2. Every wheel, having to roll without attrition, can apply only a compression force on one track, normal to the double track. Let us consider now the equilibrium of the rigid element $\mathfrak{C}$. To this aim we isolate $\mathfrak{C}$, disconnecting the double tracks from the wheels. Clearly the isolated element is submitted to

- horizontal compression forces on the wheels, made from the left double track
- vertical compression forces on the wheels, made from the right double track.

From the equilibrium of the isolated element $\mathfrak{C}$

- to horizontal translation we get that the vertical compression forces on the wheels have resultant zero
- to vertical translation we get that the horizontal compression forces on the wheels have resultant zero
- to rotation we get that the vertical compression forces on the wheels and the horizontal compression forces on the wheels have equal and opposite moments $\boldsymbol{X_1}$ (fig. 4.1.13).

Conclusively, the double sliding joint introduce 1 static unknown.

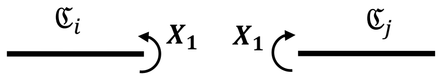

Fig. 4.1.13

From the previous analysis it follows that every constraint introduce a number of static unknowns equal to its kinematic order.

Remark 4.1.1 <u>We point out that as a rule every constraint has dimensions very small with respect to the dimensions of the structure. As a consequence, to gain clearness, we will draw them of great dimensions but in the diagrams traced on the structures we will consider them dot-like</u>. □

The mechanical load applied on the structure is supposed constituted by

- forces distributed on the one-dimensional bodies of the structure, normal to their axes and measured in *metric ton /m*
- forces concentrated in some cross sections of the one-dimensional bodies of the structure, normal to their axes and measured in *metric ton*
- couples concentrated in some cross sections of the one-dimensional bodies of the structure and measured in *metric ton* $\cdot$ *m*.

The thermal load applied on the structure is supposed constituted by a variation of the initial uniform temperature $\mathbb{T}_i$. Such variation is always constant along the beam axis and in any cross section it can

- be constant, and then equal to a uniform increment $\Delta\mathbb{T}_c$, measured in °*C*
- vary with linear law, that in any case can be decomposed in a uniform increment $\Delta\mathbb{T}_c$ and in a linear field $\mathbb{T}$, individualized by a range $\Delta\mathbb{T}$ and such that $\mathbb{T}(G) = \mathbb{T}_i$ (measured in °C).

4.1.2 The static matrix

Let us

- consider a generic one-dimensional plane structure
- denote with $t \in \mathbf{N}$ the number of its bodies
- denote with $\mathfrak{C}_1, \dots, \mathfrak{C}_t$ its bodies
- denote with $s \in \boldsymbol{N}$ the sum of the kinematic orders of the constraints
- suppose to be in the small displacements field
- assume a Cartesian orthogonal reference frame *O,x,y* (fig. 4.1.1)
- denote with $X_1, \dots, X_s$ the constraint reactions.

We write, $\forall i \in \{1, \dots, t\}$, the three cardinal equations of *Statics* concerning the beam $\mathfrak{C}_i$. This way we obtain the following system of algebraic linear equations, called *static problem*

$$b_{11}X_1 + \cdots + b_{1s}X_s = b_1$$

$$\dots \qquad (4.1.4)$$

$$b_{3t1}X_1 + \cdots + b_{3ts}X_s = b_{3t}.$$

We call *static matrix* and denote with the symbol *B* the matrix of the coefficient matrix of the system (4.1.4), that is the rectangular matrix $3t \times s$:

(4.1.5) $$B = \begin{bmatrix} b_{11} & \dots & b_{1s} \\ \dots & \dots & \dots \\ b_{3t1} & \dots & b_{3ts} \end{bmatrix}.$$

4.1.3 Kinematic-static duality

Let us

- consider a generic one-dimensional plane structure
- denote with $t \in \mathbf{N}$ the number of its bodies
- denote with $s \in \boldsymbol{N}$ the sum of the kinematic orders of the constraints
- suppose to be in the small displacements field
- assume a Cartesian orthogonal reference frame *O,x,y* (fig. 4.1.1)
- consider the kinematic matrix *C* (3.1.17) of the kinematic problem (3.1.16)
- consider the static matrix *B* (4.1.5) of the static problem (4.1.4).

We have

[4.1.2] *It results*

(4.1.6) $$B = C^T. \diamond$$

An important consequence of the [4.1.2] is the following *kinematic-static duality*. Let us denote with

- $\boldsymbol{r}$ the degree of lability of the structure
- $\boldsymbol{h}$ the degree of hyperstaticity of the structure
- r_C the rank of the kinematic matrix *C*
- r_B the rank of the kinematic matrix *B*

so that, because of the (3.1.23)

(4.1.7) $$3t - s = \boldsymbol{r} - \boldsymbol{h}\,.$$

From the [4.1.2] it immediately follows (*kinematic-static duality in the case* $\boldsymbol{r} = 0, \boldsymbol{h} = 0$)

[4.1.3] *In any isostatic structure the constraint reactions can equilibrate the load in one and only one way. Their values are all univocally individualized by the cardinal equations of Statics.*

Proof. Let us consider any isostatic structure, so that $\boldsymbol{r} = 0, \boldsymbol{h} = 0$. From this and from the (3.1.20), (3.1.21) we get

(4.1.8) $$3t = s = r_C$$

from which

(4.1.9) $$det(C) \neq 0\,.$$

Because of the (4.1.6),(4.1.9) the static problem (4.1.4) is a *Cramer* system. In fact

$$det(B) = det(C^T) = det(C) \neq 0.$$

The thesis follows. ◊

REMARK 4.1.2 The [4.1.3] implies that in any isostatic structure the constraints can and must equilibrate the load in one and only one way, that is with the unique solution of the static problem. □

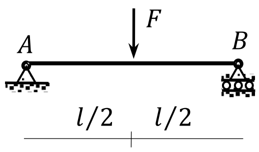

Fig. 4.1.14

PROBLEM 4.1.1 *You shall study the kinematic-static duality for the structure of fig. 4.1.14.*

Solution. The kinematic matrix of the assigned supported beam is (see problem 3.1.1)

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & l \end{bmatrix}$$

so that, since $l \in \,]0, +\infty[$

(4.1.10) $$det(C) \neq 0.$$

To build the static matrix B we remove the hinge A and the bogie B and apply their reactions (fig. 4.1.15). The three cardinal equations of *Statics* obviously are

(4.1.11) $$\begin{aligned} &X_1 = 0 \\ &X_2 + X_3 - F = 0 \\ &X_3 l - F\frac{l}{2} = 0 \end{aligned}$$

so that

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & l \end{bmatrix}.$$

We can verify that, as announced in advance by the [4.1.2], $B = C^T$. So $det(B) = det(C)$, from which, because of (4.1.10), $det(B) \neq 0$. This way we have verified that, as announced in advance by the [4.1.3], the constraint reactions of the assigned isostatic structure are univocally determined by the static problem (4.1.11). □

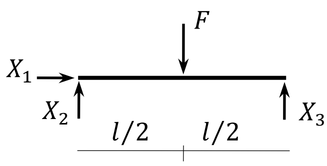

Fig. 4.1.15

From the [4.1.2] it also follows (*kinematic-static duality in the case* $\boldsymbol{r} = 0, \boldsymbol{h} > 0$)

[4.1.4] *In any hyperstatic but not labile structure the constraint reactions* *can equilibrate the load in ∞^{s-3t} ways. The cardinal equations of Statics are not enough to univocally determine the constraint reactions.*

Proof. Let us consider any structure such that $\boldsymbol{r}=0, \boldsymbol{h}>0$. From the (3.1.20) it follows

$$r_C = 3t. \tag{4.1.12}$$

We know, from the *Matrix analysis*, that every matrix and its transpose have the same rank. From this and from the (4.1.12) we get that B has rank $3t$. As a consequence a minor D of B of order $3t$ exists, such that

$$det(D) \neq 0\ . \tag{4.1.13}$$

Now we notice that the (3.1.21) and (4.1.12) imply

$$s > 3t.$$

Hence in the static problem (4.1.4) we can move on the right side of equations the columns of B not belonging to D. Because of the (4.1.13), the system obtained is a *Cramer* system. So, arbitrarily giving values to $s-3t$ static unknowns moved on the right side, we determine a solution $X_1, \dots, X_s$ of the static problem (4.1.4). The thesis follows. ◊

REMARK 4.1.3 The [4.1.4] implies that in any hyperstatic but not labile structure the constraints can equilibrate the external loads in ∞^{s-3t} ways. We underline that this result is true in the (abstract) hypothesis of rigid bodies. In reality all the bodies are deformable. Later we will see that, adding to the cardinal equations of the *Statics* opportune equations that take into account the deformability of the beams, the constraints of the structure can and must balance the load applied in one and only one way. □

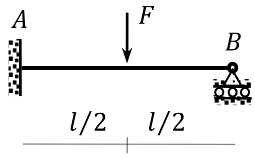

Fig. 4.1.16

PROBLEM 4.1.2 *You shall study the kinematic-static duality for the structure of fig.*

4.1.16.

Solution. The kinematic matrix of the assigned structure is, as seen in problem 3.1.6

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & l \end{bmatrix}.$$

Obviously $r_C \leq 3$. Let us consider the minor of C

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & l \end{bmatrix}.$$

Since $det(D) \neq 0$, we have $r_C = 3$. Consequently, since every matrix and its transpose have the same rank

(4.1.14) $$r_{C^T} = 3.$$

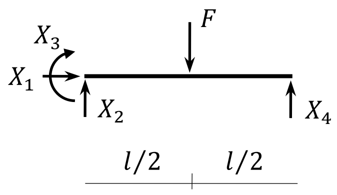

Fig. 4.1.17

To build the static matrix B we remove from the structure the fixed joint A and the bogie B and apply their reactions (fig. 4.1.17). The three cardinal equations of *Statics* obviously are

(4.1.15) $$\begin{aligned} &X_1 = 0 \\ &X_2 + X_4 - F = 0 \\ &X_3 + l\,X_4 - F\,\frac{l}{2} = 0 \end{aligned}$$

and then

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & l \end{bmatrix}.$$

We can verify that, as announced in advance by the [4.1.2], $B = C^T$. As a consequence, taking into account the (4.1.14), B has rank 3. So B admits a minor $\widetilde{D}$ of order three such that $det(\widetilde{D}) \neq 0$. Choosing

$$\widetilde{D} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we obtain from (4.1.15) the *Cramer* system

$$\begin{aligned} X_1 &= 0 \\ X_2 &= F - X_4 \\ X_3 &= F\frac{l}{2} - l\,X_4\,. \end{aligned}$$

So, arbitrarily assigning X_4, we univocally obtain X_1, X_2, X_3. So the constraints of the structure can equilibrate the load F in $\infty^{\boldsymbol{h}} = \infty^1$ ways. Clearly the obtained result only holds for structures constituted by rigid beams. □

Finally, from the [4.1.2] it also follows (*kinematic-static duality in the case* $\boldsymbol{r} > 0$)

[4.1.5] *In the general case in a labile structure the constraint reactions are not able to equilibrate the applied loads.*

Proof. Let us consider any labile structure. Because $\boldsymbol{r} > 0$, taking into account the (3.1.20) we obtain

(4.1.16) $$r_C < 3t.$$

We know, from the *Matrix analysis*, that every matrix and its transpose have the same rank. From this and from the [4.1.2] and the (4.1.16) we get that the static matrix B has rank

$$r_B < 3t.$$

In the general case the complete matrix $\tilde{B}$ of the static problem (4.1.4), that is

$$\tilde{B} = \begin{bmatrix} b_{11} & \dots & b_{1s} & b_1 \\ \dots & \dots & \dots & \dots \\ b_{3t1} & \dots & b_{3ts} & b_{3t} \end{bmatrix}$$

has rank $r_{\tilde{B}} \neq r_B$. In fact, the column of known terms is due to loads and then is not identically zero. Then, because of the *Rouché-Capelli* theorem, in the general case the static problem (4.1.4) doesn't admit solution. The thesis follows. ◊

PROBLEM 4.1.3 *You shall study the kinematic-static duality for the structure of fig. 4.1.18.*

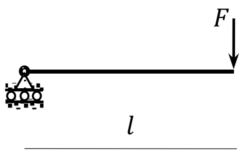

Fig. 4.1.18

Solution. The kinematic matrix of the assigned structure is, as proven in problem 3.1.7

$$C = [0 \quad 1 \quad 0]$$

and then $r_C = 1,\ \boldsymbol{r} = 2,\ \boldsymbol{h} = 0$. To build the static matrix B we remove from the structure the bogie and apply its reaction (fig. 4.1.19). The three cardinal equations of *Statics* obviously are

$$\begin{aligned} &0 = 0 \\ &X_1 - F = 0 \\ &F\,l = 0 \end{aligned} \qquad (4.1.17)$$

so that

$$B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \tilde{B} = \begin{bmatrix} 0 & 0 \\ 1 & F \\ 0 & -Fl \end{bmatrix}.$$

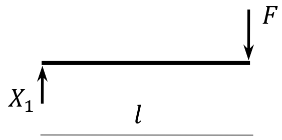

Fig. 4.1.19

Since $F \neq 0$ and $L \neq 0$ we get $r_{\tilde{B}} \neq r_B$. So the static problem (4.1.7) doesn't admit solution. So in this structure the constraints are not able to equilibrate the load. This was physically evident since the beginning. □

Occasionally the constraints of a labile structure are able of to balance the applied load. This happens when the applied load doesn't excite the release degrees of the structure. When this happens and for more $\boldsymbol{h} = 0$, the case is of particular interest. In such case, we say that the structure is *isostatic for the particular load condition*.

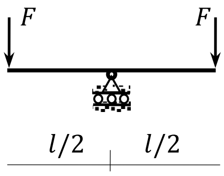

Fig. 4.1.20

PROBLEM 4.1.4 *You shall study the kinematic-static duality for the structure of fig. 4.1.20.*

Solution. We easily verify that

- the structure of fig. 4.1.20 is isostatic for the particular load condition
- the reaction of the bogie is that of fig. 4.1.21. □