Aldo Maceri

Theory of Vectors

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Aldo Maceri – *Theory of Vectors*

PREFACE

(to complete ebook)

The *structures* are solid bodies to which we ask to bear in safety the mechanical and/or thermal applied load. At times, we also wonder to them to contain some displacements within limits fixed in advance. When the load applied on the structure doesn't vary in the time, we say that it deals with a problem of *Statics*. If the load is varying in the time, we say that it deals with a problem of *Dynamics*.

This book furnishes the fundamental notions of the *Statics*. It constantly deals with the matters in exhaustive but clear way, within the classical *Physics* and the classical *Mathematical Analysis*. It mainly wants to make available to the *Engineer* all the necessary operational tools to correctly formulate and to resolve whatever problem of *Statics*, related to structures of elevated complexity also.

The study of the *Statics* begins with the exposure of the *Theory of vectors* (chap. 1). It deals with a matter of the *Mathematical Physics*, completed with a technique of graphic calculation called *funicular polygon*.

The structures are formed by one-dimensional beams connected among them and the foundation from bilateral constraints (whose physical description and analytical simulation are given in chap. 2). The basic element of any one-dimensional structure is the *thermoelastic deflected beam*. Its mathematical model is detailed in chap. 2. The linearity of such model implies two mathematical

properties, called *superposition principle* and *displacements principle*. Their proof is elementary, but they are revealed to be of *fundamental operating importance* for the execution of the calculations of the *Statics*.

The structural analysis begins with the *Kinematics* (chap. 3). The *kinematic matrix* of the structure is built. It allows to establish if

- the constraints are insufficient. In such case the structure is kinematic. It is called *labile* and the number of its release degrees is denoted with ***r***
- the constraints are strictly sufficient. In such case we say that the structure has *hyperstatic degree* ***h***=0. Any structure having ***r***=0 and ***h***=0 is called *isostatic*
- the constraints are superabundant. The sum of the orders of the superabundant constraints is denoted with ***h***. Any structure having ***r***=0 and ***h***>0 is called *hyperstatic*.

The *kinematic analysis* is completed by an operating technique called *kinematic chains*.

After that, the *Statics* begins with the building of the *static matrix,* immediately followed by the *kinematic-static duality* (chap. 4). The *duality* assures that

- in any *labile* structure the constraints cannot balance any load and then it is able to collapse. But if none of the release degree is excited from the load, the structure behaves like isostatic or hyperstatic
- in any *isostatic* structure the constraints balance the load in unique way
- in any *hyperstatic* structure the constraints can balance the load in infinite way. However, the last result is true only in the *Statics of rigid bodies*. In chap. 5 it is shown that in any real (and then deformable) *hyperstatic* structure the

constraints balance the load in unique way.

Actually, in *Statics* the *automatic structural calculus* allows to determine the state of stress and the field of displacements of any non-labile structure. The technique employed (detailed in chap. 4) is founded on the *displacements principle* and it is called *displacements method*.

But *inevitably* any automatic structural calculation does approximations that sometimes have produced output containing dangerous errors. To avoid such errors, the *structural Engineer* must know *also* the other techniques of calculation of the *Statics*, briefly called *forces method*.

In any *isostatic* structure, to calculate the state of stress the *forces method* can employ *only* considerations of equilibrium (chap. 5). The *graphic Statics* is a useful additional tool.

In any *hyperstatic* structure, to calculate the state of stress the *forces method* must *also* employ considerations of congruence or the *virtual works principle* (chap. 6).

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CHAPTER 1

THEORY OF VECTORS

1.1 Introduction

1.1.1 Linear spaces on $\mathfrak{R}$ [1.1.1]

We call *linear space* on $\mathfrak{R}$ or *real linear space* or *vector space* on $\mathfrak{R}$ a set $\boldsymbol{V}$

- provided of a law (called *addition*) that associates to each element $(\boldsymbol{x},\boldsymbol{y})$ of $\boldsymbol{V}\mathbf{x}\boldsymbol{V}$ an element $\boldsymbol{x}+\boldsymbol{y}$ of $\boldsymbol{V}$ and has the properties

  - (1.1.1) $\forall \boldsymbol{x},\boldsymbol{y} \in \boldsymbol{V}$ $\boldsymbol{x}+\boldsymbol{y}=\boldsymbol{y}+\boldsymbol{x}$ (*commutative*)
  - (1.1.2) $\forall \boldsymbol{x},\boldsymbol{y},\boldsymbol{z} \in \boldsymbol{V}$ $\boldsymbol{x}+(\boldsymbol{y}+\boldsymbol{z})=(\boldsymbol{x}+\boldsymbol{y})+\boldsymbol{z}$ (*associative*)
  - (1.1.3) it exists an element of $\boldsymbol{V}$, called *zero* and denoted with $\mathbf{0}$, such that $\forall \boldsymbol{x} \in \boldsymbol{V}$ $\boldsymbol{x}+\mathbf{0}=\boldsymbol{x}$
  - (1.1.4) $\forall \boldsymbol{x} \in \boldsymbol{V}$ it exists an element of $\boldsymbol{V}$, denoted with $-\boldsymbol{x}$, such that $-\boldsymbol{x}+\boldsymbol{x}=\mathbf{0}$

- provided of a law (called *multiplication*) that associates to each element $(\lambda,\boldsymbol{x})$ of $\mathfrak{R}\mathrm{x}\boldsymbol{V}$ an element of $\boldsymbol{V}$, denoted $\lambda\boldsymbol{x}$ and called product of $\boldsymbol{x}$ for the *scalar* λ

- provided of the properties

[1.1.1] We denote with $\mathfrak{R}$ the set of the real numbers.

- (1.1.5) $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ e $\forall \lambda \in \mathfrak{R}$ $\lambda(\boldsymbol{x}+\boldsymbol{y}) = \lambda\boldsymbol{x} + \lambda\boldsymbol{y}$ (*distributive*)
- (1.1.6) $\forall \boldsymbol{x} \in \boldsymbol{V}$ e $\forall \lambda, \mu \in \mathfrak{R}$ $(\lambda+\mu)\boldsymbol{x} = \lambda\boldsymbol{x} + \mu\boldsymbol{x}$ (*distributive*)
- (1.1.7) $\forall \boldsymbol{x} \in \boldsymbol{V}$ e $\forall \lambda, \mu \in \mathfrak{R}$ $(\lambda\mu)\boldsymbol{x} = \lambda(\mu\boldsymbol{x})$
- (1.1.8) $\forall \boldsymbol{x} \in \boldsymbol{V}$ $1\boldsymbol{x} = \boldsymbol{x}$.

The relations from (1.1.1) to (1.1.8) are said *axioms of a linear space*. The elements of $\boldsymbol{V}$ are also called ***vectors***.

REMARK 1.1.1 Using the axioms from (1.1.1) to (1.1.8) we easily prove that a linear space $\boldsymbol{V}$ has the properties

- $\forall \boldsymbol{x} \in \boldsymbol{V}$ $0\boldsymbol{x} = \boldsymbol{0}$
- $\forall \boldsymbol{x} \in \boldsymbol{V}$ e $\forall \lambda \in \mathfrak{R}$ $(-\lambda)\boldsymbol{x} = \lambda(-\boldsymbol{x}) = -(\lambda\boldsymbol{x})$
- $\forall \lambda \in \mathfrak{R}$ $\lambda\boldsymbol{0} = \boldsymbol{0}$
- $\forall \boldsymbol{x} \in \boldsymbol{V} - \{\boldsymbol{0}\}$ $(\lambda\boldsymbol{x} = \boldsymbol{0}) \Rightarrow (\lambda = 0)$
- $\forall \boldsymbol{x} \in \boldsymbol{V} - \{\boldsymbol{0}\}$ e $\forall \lambda, \mu \in \mathfrak{R}$ $(\lambda\boldsymbol{x} = \mu\boldsymbol{x}) \Rightarrow (\lambda = \mu)$
- $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$ e $\forall \lambda \in \mathfrak{R} - \{0\}$ $(\lambda\boldsymbol{x} = \lambda\boldsymbol{y}) \Rightarrow (\boldsymbol{x} = \boldsymbol{y})$
- $\forall \boldsymbol{x} \in \boldsymbol{V}$ $-(-\boldsymbol{x}) = \boldsymbol{x}$
- $\forall \boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{V}$, denoting with $\boldsymbol{x} - \boldsymbol{y}$ the element $\boldsymbol{x} + (-\boldsymbol{y})$, we have $\boldsymbol{x} - (-\boldsymbol{y}) = \boldsymbol{x} + \boldsymbol{y}$. □

REMARK 1.1.2 A linear space is never empty, since it always has the element $\boldsymbol{0}$. □

Let $\boldsymbol{x_1}, \dots, \boldsymbol{x_n}$ [resp. $\lambda_1, \dots, \lambda_n$] be $n \in \boldsymbol{N}$ [1.1.2] elements of a linear space $\boldsymbol{V}$. By successive operations we define the element

$$\Big(\big((\lambda_1\boldsymbol{x_1} + \lambda_2\boldsymbol{x_2}) + \lambda_3\boldsymbol{x_3}\big) + \cdots\Big) + \lambda_n\boldsymbol{x_n} \in \boldsymbol{V}.$$

This element is denoted

[1.1.2] We denote with $\boldsymbol{N}$ the set of the positive integer numbers.

$$\lambda_1 \boldsymbol{x_1} + \cdots + \lambda_n \boldsymbol{x_n} = \sum_{i=1}^{n} \lambda_i \boldsymbol{x_i}$$

and called *finite linear combination of the elements* $\boldsymbol{x_1}, \dots, \boldsymbol{x_n}$ *with coefficients* $\lambda_1, \dots, \lambda_n$ *in* $\mathfrak{R}$. From axioms (1.1.1) and (1.1.2) it obviously follows that executing the successive operations in another order or gathering some operations, the same element $\sum_{i=1}^{n} \lambda_i \boldsymbol{x_i}$ is gotten.

REMARK 1.1.3 We easily prove that a not empty subset $\boldsymbol{Q}$ of a linear space $\boldsymbol{V}$ on $\mathfrak{R}$, satisfying the relations

$$(\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{Q}) \Rightarrow (\boldsymbol{x} + \boldsymbol{y} \in \boldsymbol{Q})$$
$$(\boldsymbol{x} \in \boldsymbol{Q} \text{ and } \lambda \in \mathfrak{R}) \Rightarrow (\lambda \boldsymbol{x} \in \boldsymbol{Q})$$

is a linear space on $\mathfrak{R}$. $\boldsymbol{Q}$ is said *linear subspace of* $\boldsymbol{V}$ *on* $\mathfrak{R}$. □

It is said that a linear space $\boldsymbol{V}$ on $\mathfrak{R}$ is an *ordered linear space* if $\boldsymbol{V}$ is an ordered set [1.1.3] and the following conditions are satisfied

- (1.1.9) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \boldsymbol{V} \quad (\boldsymbol{x} \leq \boldsymbol{y}) \Rightarrow (\boldsymbol{x} + \boldsymbol{z} \leq \boldsymbol{y} + \boldsymbol{z})$
- (1.1.10) $\forall \boldsymbol{x} \in \boldsymbol{V}$ e $\forall \lambda \in \mathfrak{R} \quad (\boldsymbol{x} \geq \boldsymbol{0} \text{ and } \lambda \geq 0) \Rightarrow (\lambda \boldsymbol{x} \geq \boldsymbol{0})$.

Let

- $\boldsymbol{V}$ be a linear space on $\mathfrak{R}$
- $n \in \boldsymbol{N}$
- $\forall i \in \{1, \dots, n\} \quad \boldsymbol{v_i} \in \boldsymbol{V}$.

It is said that the elements $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ are *linearly dependent* if n real

---

[1.1.3] A set V is said *ordered* if an order relation in V exists. It is said that a relation $\leq$ in a set V is an *order relation in* V if it is *antisymmetric* (that is $\forall x, y \in V \quad (x \leq y \text{ and } y \leq x) \Rightarrow (x = y)$), *transitive* (that is $\forall x, y, z \in V \quad (x \leq y \text{ and } y \leq z) \Rightarrow (x \leq z)$) and *reflexive* (that is $\forall x \in V \quad x \leq x$).

numbers $\lambda_1, \dots, \lambda_n$ exist such that $(\lambda_1, \dots, \lambda_n) \neq (0, \dots, 0)$ and

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{v_i} = \mathbf{0}\,.$$

Let

- $\boldsymbol{V}$ be a linear space on $\mathfrak{R}$
- $n \in \boldsymbol{N}$
- $\forall i \in \{1, \dots, n\} \quad \boldsymbol{v_i} \in \boldsymbol{V}$.

It is said that the elements $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ are *linearly independent* if $\forall(\lambda_1, \dots, \lambda_n) \in \mathfrak{R}^n - (0, \dots, 0)$ it results

$$\sum_{i=1}^{n} \lambda_i \boldsymbol{v_i} \neq \mathbf{0}\,.$$

REMARK 1.1.4 Obviously, the element **0**=(0,...,0) is linearly dependent and any $\boldsymbol{v} \in \boldsymbol{V} - \{\mathbf{0}\}$ is linearly independent. □

REMARK 1.1.5 Let $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ be *n* linearly dependent elements of a real linear space ***V***. It is immediate to prove that one of them is linear combination of the remaining. □

Let

- $\boldsymbol{V}$ be a linear space on $\mathfrak{R}$
- $n \in \boldsymbol{N}$
- $\forall i \in \{1, \dots, n\} \quad \boldsymbol{v_i} \in \boldsymbol{V}$.

We said that the subset $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ is a *basis for **V** of dimension n* if

- the elements $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ are linearly independent
- $\forall \boldsymbol{v} \in \boldsymbol{V} \quad \exists(\lambda_1, \dots, \lambda_n) \in \mathfrak{R}^n$ such that $\boldsymbol{v} = \sum_{i=1}^{n} \lambda_i \boldsymbol{v_i}$.

Let $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ be a *basis of dimension n* for a real linear space $\boldsymbol{V}$. Since the elements $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$ are linearly independent, it is easy to prove that every element $\boldsymbol{v} \in \boldsymbol{V}$ can be expressed in only way as linear combination of $\boldsymbol{v_1}, \dots, \boldsymbol{v_n}$. So, $\forall i \in \{1, \dots, n\}$ we call λ_i *i*th *component of* $\boldsymbol{v}$ *in the basis* $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$.

Let $\{\boldsymbol{v_1}, \dots, \boldsymbol{v_n}\}$ be a *basis of dimension n* for a real linear space $\boldsymbol{V}$. It is possible to prove that every other basis of $\boldsymbol{V}$ is finite and has dimension *n*. That's the reason why if a real linear space $\boldsymbol{V}$ admits a basis of dimension *n* we say that $\boldsymbol{V}$ is a real linear space of *dimension n*.

1.1.2 The linear space $\mathfrak{R}^n$

Let us consider now any $n \in \boldsymbol{N}$ and the set

$$\mathfrak{R}^n = \{(x_1, \dots, x_n) : \forall i \in \{1, \dots, n\} \quad x_i \in \mathfrak{R}\}.$$

We immediately underline that it furnishes the most important example of real linear space of dimension *n*. To this aim we provide $\mathfrak{R}^n$ of the law + (*addition*)

$$(1.1.11) \qquad \forall\big(\boldsymbol{x} = (x_1, \dots, x_n), \boldsymbol{y} = (y_1, \dots, y_n)\big) \in \mathfrak{R}^n \times \mathfrak{R}^n \rightarrow$$
$$\rightarrow \boldsymbol{x} + \boldsymbol{y} = (x_1 + y_1, \dots, x_n + y_n) \in \mathfrak{R}^n$$

and of the law · (*multiplication*)

$$(1.1.12) \qquad \forall\big(\lambda, \boldsymbol{x} = (x_1, \dots, x_n)\big) \in \mathfrak{R} \times \mathfrak{R}^n \rightarrow$$
$$\rightarrow \lambda \cdot \boldsymbol{x} = \lambda\, \boldsymbol{x} = (\lambda x_1, \dots, \lambda x_n) \in \mathfrak{R}^n.$$

It is immediate to verify that $\mathfrak{R}^n$, provided of such laws, satisfies the axioms from (1.1.1) to (1.1.8). Insofar $\mathfrak{R}^n$ is real linear space.

Evidently the zero of $\mathfrak{R}^n$ is $\mathbf{0}$ = (0,...,0) and

- $\forall\lambda \in \mathfrak{R} \qquad \lambda\mathbf{0} = \mathbf{0}$
- $\forall\boldsymbol{x} \in \mathfrak{R}^n \qquad 0\boldsymbol{x} = \mathbf{0}$
- $\forall\lambda \in \mathfrak{R}$ and $\forall\boldsymbol{x} \in \mathfrak{R}^n \quad (-\lambda)\boldsymbol{x} = \lambda(-\boldsymbol{x}) = -(\lambda\boldsymbol{x})$
- $\forall\lambda \in \mathfrak{R}$ and $\forall\boldsymbol{x} \in \mathfrak{R}^n - \{\mathbf{0}\} \qquad (\lambda\boldsymbol{x} = \mathbf{0}) \Rightarrow (\lambda = 0)$
- $\forall\lambda, \mu \in \mathfrak{R}$ and $\forall\boldsymbol{x} \in \mathfrak{R}^n - \{\mathbf{0}\} \qquad (\lambda\boldsymbol{x} = \mu\boldsymbol{x}) \Rightarrow (\lambda = \mu)$
- $\forall\lambda \in \mathfrak{R} - \{0\}$ and $\forall\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad (\lambda\boldsymbol{x} = \lambda\boldsymbol{y}) \Rightarrow (\boldsymbol{x} = \boldsymbol{y})$.

Let us consider the vectors of $\mathfrak{R}^n$

$$\begin{gathered}\boldsymbol{e_1} = (1,0,0,\dots,0)\\ \boldsymbol{e_2} = (0,1,0,\dots,0)\\ \dots\dots\dots\dots\dots\dots\dots\\ \boldsymbol{e_n} = (0,0,\dots,0,1).\end{gathered}$$

It is easy to prove that $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ is a basis of dimension *n* for the real linear space $\mathfrak{R}^n$. $\{\boldsymbol{e_1}, \dots, \boldsymbol{e_n}\}$ is called *natural basis*. In particular we have

$$(1.1.13) \qquad \forall\boldsymbol{x} = (x_1, \dots, x_n) \in \mathfrak{R}^n \qquad \boldsymbol{x} = \sum_{i=1}^{n} x_i \boldsymbol{e_i}\,.$$

We call *Euclidean distance* between two any points $\boldsymbol{x} = (x_1, \dots, x_n)$, $\boldsymbol{y} = (y_1, \dots, y_n)$ of $\mathfrak{R}^n$, and denote it with the symbol $dist(\boldsymbol{x}, \boldsymbol{y})$, the not negative real number

$$\sqrt{(y_1 - x_1)^2 + \dots + (y_n - x_n)^2}.$$

Obviously, it results

$$(1.1.14) \qquad \forall\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad dist(\boldsymbol{x}, \boldsymbol{y}) = dist(\boldsymbol{y}, \boldsymbol{x})$$

$$(1.1.15) \qquad \forall\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad (dist(\boldsymbol{x}, \boldsymbol{y}) = 0) \Leftrightarrow (\boldsymbol{x} = \boldsymbol{y})$$

$$(1.1.16) \qquad \forall\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathfrak{R}^n \qquad dist(\boldsymbol{x} + \boldsymbol{z}, \boldsymbol{y} + \boldsymbol{z}) = dist(\boldsymbol{x}, \boldsymbol{y})$$

(1.1.17) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n$ and $\forall \lambda \in \mathfrak{R} \quad dist(\lambda \boldsymbol{x}, \lambda \boldsymbol{y}) = |\lambda| dist(\boldsymbol{x}, \boldsymbol{y})$.

Let us prove now the following important property, called *triangle property* of the *Euclidean distance*:

[1.1.1] *It results*

(1.1.18) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathfrak{R}^n \qquad dist(\boldsymbol{x}, \boldsymbol{z}) \leq dist(\boldsymbol{x}, \boldsymbol{z}) + dist(\boldsymbol{z}, \boldsymbol{y})$.

Proof. We must prove that for any $(x_1, \dots, x_n), (y_1, \dots, y_n), (z_1, \dots, z_n) \in \mathfrak{R}^n$ it results

(1.1.19) $\sqrt{\sum_{i=1}^n (y_i - x_i)^2} \leq \sqrt{\sum_{i=1}^n (z_i - x_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$.

To this aim we put

(1.1.20) $\forall i \in \{1, \dots, n\} \quad a_i = z_i - x_i\,, \;\; b_i = (y_i - z_i)$

and as first step we prove that

(1.1.21) $$\left(\sum_{i=1}^n a_i b_i\right)^2 \leq \left(\sum_{i=1}^n {a_i}^2\right)\left(\sum_{i=1}^n {b_i}^2\right).$$

The (1.1.21) is banally true if $\forall i \in \{1, \dots, n\} \quad a_i = 0$. In the other possible case it results $\sum_{i=1}^n {a_i}^2 > 0$. As a consequence, putting $a = \sum_{i=1}^n {a_i}^2$, it results $a > 0$. Moreover, we put $b = \sum_{i=1}^n a_i b_i$, $c = \sum_{i=1}^n {b_i}^2$ and consider the obvious relation

$$\forall \lambda \in \mathfrak{R} \qquad \left(\sum_{i=1}^n (\lambda a_i - b_i)^2\right) \geq 0$$

from which

$$\forall\lambda \in \mathfrak{R} \qquad \left(\sum_{i=1}^{n} {a_i}^2\right)\lambda^2 - 2\left(\sum_{i=1}^{n} a_i b_i\right)\lambda + \sum_{i=1}^{n} {b_i}^2 \geq 0$$

from which

$$\forall\lambda \in \mathfrak{R} \qquad a\,\lambda^2 - 2\,b\lambda + c \geq 0$$

from which, since $a > 0$

$$\forall\lambda \in \mathfrak{R} \qquad a\left[\left(\lambda - \frac{b}{a}\right)^2 + \frac{ac - b^2}{a^2}\right] \geq 0$$

from which, since $a > 0$, it results $ac - b^2 \geq 0$. So, the (1.1.21) is true also in the case $\sum_{i=1}^{n} {a_i}^2 > 0$.

From (1.1.21) we immediately have

$$2\sum_{i=1}^{n} a_i b_i \leq 2\sqrt{\left(\sum_{i=1}^{n} {a_i}^2\right)}\sqrt{\left(\sum_{i=1}^{n} {b_i}^2\right)}$$

from which

$$\text{(1.1.22)} \qquad \sum_{i=1}^{n} (a_i + b_i)^2 \leq \left(\sqrt{\sum_{i=1}^{n} {a_i}^2} + \sqrt{\sum_{i=1}^{n} {b_i}^2}\right)^2$$

from which, taking into account the (1.1.20), the thesis follows. ◊

Let $\boldsymbol{x} = (x_1, \dots, x_n), \boldsymbol{y} = (y_1, \dots, y_n)$ be any vectors of $\mathfrak{R}^n$. We call *scalar product* of $\boldsymbol{x}$ and $\boldsymbol{y}$, which we denote with $\boldsymbol{x} \times \boldsymbol{y}$, the real number

$$\text{(1.1.23)} \qquad \boldsymbol{x} \times \boldsymbol{y} = x_1 y_1 + \cdots + x_n y_n\,.$$

The following relations are obvious

(1.1.24) $\forall \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathfrak{R}^n \qquad (\boldsymbol{x} + \boldsymbol{y}) \times \boldsymbol{z} = \boldsymbol{x} \times \boldsymbol{z} + \boldsymbol{y} \times \boldsymbol{z}$

(1.1.25) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad \boldsymbol{x} \times \boldsymbol{y} = \boldsymbol{y} \times \boldsymbol{x}$

(1.1.26) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n$ and $\forall \lambda \in \mathfrak{R} \qquad (\lambda \boldsymbol{x}) \times \boldsymbol{y} = \lambda(\boldsymbol{x} \times \boldsymbol{y})$ **.**

Let $\boldsymbol{x} = (x_1, \dots, x_n)$ be any point of $\mathfrak{R}^n$. We call *Euclidean norm* of $\boldsymbol{x}$, and denote with $\|\boldsymbol{x}\|$, the not negative real number

(1.1.27) $$\|\boldsymbol{x}\| = \sqrt{{x_1}^2 + \cdots + {x_n}^2}$$

and then

(1.1.28) $\forall \boldsymbol{x} \in \mathfrak{R}^n \qquad \|\boldsymbol{x}\|^{\mathbf{2}} = \boldsymbol{x} \times \boldsymbol{x}$.

From the (1.1.15) we immediately get

(1.1.29) $\forall \boldsymbol{x} \in \mathfrak{R}^n \qquad (\|\boldsymbol{x}\| = 0) \Leftrightarrow (\boldsymbol{x} = \mathbf{0})$.

From the (1.1.17) we immediately get

(1.1.30) $\forall \boldsymbol{x} \in \mathfrak{R}^n$ and $\forall \lambda \in \mathfrak{R} \qquad \|\lambda \boldsymbol{x}\| = \lambda \|\boldsymbol{x}\|$.

Moreover

[1.1.2] *It results*

(1.1.31) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad \|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$.

Proof. In the proof of [1.1.1] we have obtained the (1.1.22) with a reasoning that is valid for any $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathfrak{R}^n$. The thesis

follows. ◊

Finally, the following inequality of *Cauchy-Schwarz* subsists

[1.1.3] *It results*

(1.1.32) $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n \qquad |\boldsymbol{x} \times \boldsymbol{y}| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|.$

Proof. In the proof of [1.1.1] we have obtained the (1.1.21) with a reasoning that is valid for any $(a_1, \dots, a_n), (b_1, \dots, b_n) \in \mathfrak{R}^n$. The thesis follows. ◊

Clearly any $\boldsymbol{x} = (x_1, \dots, x_n) \in \mathfrak{R}^n$ individualizes in the *n*-dimensional space both a *point* $\boldsymbol{P}$ and the *vector* having first end $\mathbf{0}$ and second end $\boldsymbol{P}$. Since the (1.1.13), in the *Cartesian orthogonal frame of reference* $(\mathbf{0}, \boldsymbol{e_1}, \dots, \boldsymbol{e_n})$, $\forall i \in \{1, \dots, n\}$ x_i denotes both the *i*th *orthogonal coordinate* of the point $\boldsymbol{P}$ and the *i*th *orthogonal component* of the vector $\mathbf{0}\boldsymbol{P}$. *For this motive the elements of* $\mathfrak{R}^n$ *are indifferently called* ***points*** *or* ***vectors***.

Remark 1.1.6 Let $\boldsymbol{x}, \boldsymbol{y} \in \mathfrak{R}^n$. When we consider $\|\boldsymbol{x}\|$, we call $\boldsymbol{x}$ vector. When we consider $dist(\boldsymbol{x}, \boldsymbol{y})$, we call $\boldsymbol{x}, \boldsymbol{y}$ points. When we consider $\boldsymbol{x} \times \boldsymbol{y}$, we call $\boldsymbol{x}, \boldsymbol{y}$ vectors. □

Let $\boldsymbol{x}$ be any point of $\mathfrak{R}^n$, $\boldsymbol{y}$ be any vector of $\mathfrak{R}^n - \{\mathbf{0}\}$. The subset of $\mathfrak{R}^n$

(1.1.33) $r = \{\boldsymbol{z} \in \mathfrak{R}^n : \boldsymbol{z} = \boldsymbol{x} + \lambda \boldsymbol{y},\ \lambda \in \mathfrak{R}\}$

is called *straight line of* $\mathfrak{R}^n$ *passing through the point* $\boldsymbol{x}$ *and parallel to vector* $\boldsymbol{y}$. The components of $\boldsymbol{y}$ are said *direction numbers* of *r*.
If $\|\boldsymbol{y}\| = 1$, the vector $\boldsymbol{y}$ is said *unit vector* of *r* and his components are said *direction cosines* of *r*.

The (1.1.33) is said *parametric equation* of *r*. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are two any distinct points of $\mathfrak{R}^n$, another parametric equation of *r* is

$$r = \{\boldsymbol{z} \in \mathfrak{R}^n\colon\ \boldsymbol{z} = \boldsymbol{x} + \lambda(\boldsymbol{y} - \boldsymbol{x}), \qquad \lambda \in \mathfrak{R}\}.$$

1.2 Vectors of $\mathfrak{R}^3$

1.2.1 Basics of *Analytical Geometry* of the space

Let *r* be a straight line of the three-dimensional space. We choose for *r* an orientation and one of its points as origin. It is known that it is possible to build a bijective mapping between the set of the points of *r* and $\mathfrak{R}$. In it every point $A \in r$ individualizes a real number a_1 that we call *coordinate* (or *abscissa*) of A and every real number is the abscissa of one and only one point of *r*.

We consider three oriented axes x_1, x_2, x_3 passing through a point O of the three-dimensional space and two by two orthogonal. We call the system O, x_1, x_2, x_3 *Cartesian orthogonal frame of reference of origin* O *and of coordinated axes* x_1, x_2, x_3 (fig. 1.2.1).

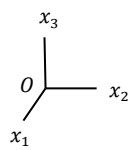

Fig. 1.2.1

Let *P* be a point of the three-dimensional space. We consider the plane α_P [resp. β_P] [resp. γ_P] passing through *P* and normal to x_1 [resp. x_2] [resp. x_3]. We denote with P_1 [resp. P_2] [resp. P_3] the point of x_1 [resp. x_2] [resp. x_3] belonging to α_P [resp. β_P] [resp. γ_P] and with p_1 [resp. p_2] [resp. p_3] the abscissa of P_1 [resp. P_2] [resp. P_3] on x_1 [resp. x_2] [resp. x_3]. We call p_1 [resp. p_2] [resp. p_3] *first* [resp. *second*] [resp. *third*] *Cartesian orthogonal coordinate* of *P*.

Let us consider a Cartesian orthogonal frame of reference O, x_1, x_2, x_3 and the mapping *f* that to every point *P* of the three-dimensional space associates the ordered triplet of real numbers (p_1, p_2, p_3) constituted by the Cartesian orthogonal coordinates of *P*. Clearly, *f* is a bijective function with values in $\mathfrak{R}^3$ and defined in the three-dimensional space. Therefore, we can indifferently denote with *P* a point of the three-dimensional space or a point (p_1, p_2, p_3) of $\mathfrak{R}^3$. In this sense we write $P = (p_1, p_2, p_3)$.

Let $A = (a_1, a_2, a_2)$, $B = (b_1, b_2, b_3)$ be two points of the three-dimensional space. We call *distance between A and B* or *length of the segment* AB, and denote with the symbol $dist(A, B)$, the non-negative real number

(1.2.1) $$dist(A, B) = ((b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2)^{\frac{1}{2}} .$$

Obviously

(1.2.2) $$(A = B) \Leftrightarrow (b_1 = a_1 , b_2 = a_2 , b_3 = a_3) \Leftrightarrow$$
$$\Leftrightarrow (dist(A, B) = 0)$$

(1.2.3) $$(A \neq B) \Leftrightarrow (dist(A, B) > 0).$$

Let $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$ be two distinct points of $\mathfrak{R}^3$. Since the (1.2.3) we can consider the real numbers

(1.2.4) $$r_1 = \frac{b_1 - a_1}{dist(A,B)} , r_2 = \frac{b_2 - a_2}{dist(A,B)} , r_3 = \frac{b_3 - a_3}{dist(A,B)} .$$

Evidently

(1.2.5) $$r_1^2 + r_2^2 + r_3^2 = 1$$

and then r_1 , r_2 , r_3 *are not zero all three of them.*

Let us consider now the subset of $\mathfrak{R}^3$

(1.2.6) $$r = \{(a_1 + t\, r_1, a_2 + t\, r_2, a_3 + t\, r_3) \in \mathfrak{R}^3 : t \in \mathfrak{R}\}\,.$$

Since $t_A = 0 \in \mathfrak{R}$ [resp. $t_B = dist(A,B) \in \mathfrak{R}$] we have $A \in r$ [resp. $B \in r$]. So, we call the subset r *oriented straight line of* $\mathfrak{R}^3$ *individualized by points A and B* or *oriented straight line of* $\mathfrak{R}^3$ *passing through points A and B*. We call *natural orientation of r* the orientation from A toward B. If $P = (p_1, p_2, p_3) \in r$, $\exists\, t_P \in \mathfrak{R}$ such that $p_1 = a_1 + t_P\, r_1,\ p_2 = a_2 + t_P\, r_2, p_3 = a_3 + t_P\, r_3$. From this and from (1.2.5) we get $dist(P,A) = |t_P|$. Moreover if $t_P > 0$ then P follows A in the natural orientation of r.

We call the number r_1 [resp. r_2] [resp. r_3] *first* [resp. *second*] [resp. *third*] *direction cosine* of *r*. Moreover, we call the number $b_1 - a_1$ [resp. $b_2 - a_2$] [resp. $b_3 - a_3$] *first* [resp. *second*] [resp. *third*] *direction number* of r.

REMARK 1.2.1 *r* is also called *oriented straight line of* $\mathfrak{R}^3$ *passing through point A and having direction cosines* r_1, r_2, r_3 . □

REMARK 1.2.2 We choose on the x_1 [resp. x_2] [resp. x_3] axis of the reference frame the points (0,0,0) and (1,0,0) [resp. (0,1,0)] [resp. (0,0,1)]. From the (1.2.4) we immediately obtain that the direction cosines of the axes x_1, x_2, x_3 of the reference frame are,

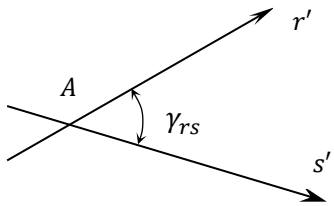

Fig. 1.2.2

respectively, (1,0,0), (0,1,0), (0,0,1). □

REMARK 1.2.3 Every couple of distinct points of r furnishes for r the same direction cosines. In fact, let us suppose that $A = (a_1, a_2, a_3)$, $B = (b_1, b_2, b_3)$, $C = (c_1, c_2, c_3)$, $D = (d_1, d_2, d_3)$ are points of r, with $A \neq B$ and $C \neq D$. As a consequence, $\exists\, t_C \in \mathfrak{R}$ and $\exists\, t_D \in \mathfrak{R}$ such that

$$c_1 = a_1 + t_C\, r_1\ , \qquad c_2 = a_2 + t_C\, r_2\ , \qquad c_3 = a_3 + t_C\, r_3$$
$$d_1 = a_1 + t_D\, r_1\ , \qquad d_2 = a_2 + t_D\, r_2\ , \qquad d_3 = a_3 + t_D\, r_3\ .$$

From this and (1.2.5) we get

$$dist(C, D) = |t_C - t_D|$$
$$d_1 - c_1 = (t_D - t_C)\, r_1\ , \quad d_2 - c_2 = (t_D - t_C)\, r_2\ , \qquad d_3 - c_3 = (t_D - t_C)\, r_3\ .$$

Since $C \neq D$ it results $t_D - t_C \neq 0$ and if D follows C in the natural orientation of r it results $t_D - t_C > 0$. As a consequence

$$\frac{b_1 - a_1}{dist(A,B)} = \frac{d_1 - c_1}{dist(C,D)}\ , \quad \frac{b_2 - a_2}{dist(A,B)} = \frac{d_2 - c_2}{dist(C,D)}\ , \quad \frac{b_3 - a_3}{dist(A,B)} = \frac{d_3 - c_3}{dist(C,D)}\ . \ \square$$

Let r and s be two any oriented straight lines of $\mathfrak{R}^3$ and A be any point of $\mathfrak{R}^3$. We denote with r' [resp. s'] the straight line parallel to r [resp. s] and passing through A. We call *angle between r and s,* and denote with the symbol γ_{rs}, the angle between r' and s' (fig. 1.2.2).

REMARK 1.2.4 Obviously $\gamma_{rs} = \gamma_{sr}$. □

Let A be any point of $\mathfrak{R}^3$. Let r be any oriented straight line of $\mathfrak{R}^3$ passing through A and having direction cosines r_1, r_2, r_3. We call *unit vector* of the oriented line r, and denote with the symbol $\boldsymbol{r}$, the oriented segment $\boldsymbol{AB}$, in which

- B is a point of r
- B follows A in the natural orientation of r

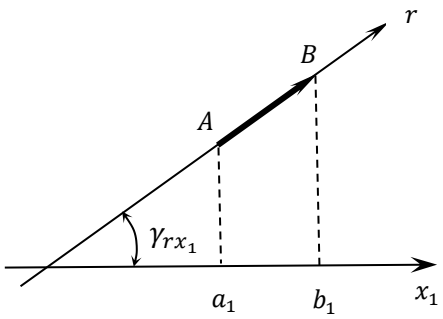

Fig. 1.2.3

- $dist(A, B) = 1$.

We observe that the orthogonal projection of $\boldsymbol{r}$ on the x_1 [resp. x_2] [resp. x_3] axis has length $b_1 - a_1$ [resp. $b_2 - a_2$] [resp. $b_3 - a_3$].

On the other hand, since $dist(A, B) = 1$, we obviously have $b_1 - a_1 = cos\,\gamma_{rx_1}$ [resp. $b_2 - a_2 = cos\,\gamma_{rx_2}$] [resp. $b_3 - a_3 = cos\,\gamma_{rx_3}$(fig. 1.2.3). As a consequence, *the direction cosine r_1 [resp. r_2] [resp. r_3] is the orthogonal component of* $\boldsymbol{r}$ *on the x_1 [resp. x_2] [resp. x_3] axis*, that is

(1.2.7) $$r_1 = cos\,\gamma_{rx_1}\ \ , r_y = cos\,\gamma_{rx_2}\ \ ,\ r_3 = cos\,\gamma_{rx_3}.$$

Let us observe now that

[1.2.1] *Let r [resp. s] be any straight line of $\mathfrak{R}^3$ having direction cosines r_1, r_2, r_3 [resp. s_1, s_2, s_3] . It results*

(1.2.8) $$cos\,\gamma_{rs} = r_1 s_1 + r_2 s_2 + r_3 s_3.$$

Proof. Let us consider the unit vector $\boldsymbol{r}$ of r. Since $\boldsymbol{r}$ is a segment of

length 1, its orthogonal projection on *s* has length $\cos\gamma_{rs}$. On the other hand, such orthogonal projection is just the sum of the orthogonal projections on *s* of the orthogonal components of $\boldsymbol{r}$ on the x_1, x_2, x_3 axes, and then

$$\cos\gamma_{rs} = r_1 \cos\gamma_{sx_1} + r_2 \cos\gamma_{sx_2} + r_3 \cos\gamma_{sx_3} .$$

Taking into account the (1.2.7), the thesis follows. ◊

Because of (1.2.6) any straight line *r* of $\mathfrak{R}^3$ can be individualized by a point *A* of $\mathfrak{R}^3$ and a triplet r_1, r_2, r_3 of real numbers satisfying the (1.2.5). Evidently the triplet r_1, r_2, r_3 assign the direction and the orientation of *r*. From (1.2.6) it immediately follows that the point *A* and the triplet $-r_1, -r_2, -r_3$ individualize a straight line of $\mathfrak{R}^3$ coinciding with *r* but having opposite orientation.

Let *r* [resp. *s*] be a straight line of $\mathfrak{R}^3$, r_1, r_2, r_3 [resp. s_1, s_2, s_3] be the direction cosines of *r* [resp. *s*]. We say that *r* and *s* are *parallel* if

$$r_1 = s_1 , \ r_2 = s_2 , \ r_3 = s_3$$

or

$$r_1 = -s_1 , \ r_2 = -s_2 , \ r_3 = -s_3 .$$

Let *r*, *s* be two any parallel lines of $\mathfrak{R}^3$. Obviously or *r* and *s* don't have in common any point or they have in common at least a point and then are coincident. In any case, obviously $\gamma_{rs} = 0$.

Let r, s be two any orthogonal straight lines of $\mathfrak{R}^3$*, having respectively direction cosines* r_1, r_2, r_3 *and* s_1, s_2, s_3*. Since the (1.2.8), it results*

$$r_1 s_1 + r_2 s_2 + r_3 s_3 = 0 \tag{1.2.9}$$

and vice versa.

Finally, let

- $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3)$ be two distinct points of $\mathfrak{R}^3$,
- $\boldsymbol{F}$ be the oriented segment of $\mathfrak{R}^3$ having first [resp. second] end A [resp. B],
- F_1 [resp. F_2] [resp. F_3] be the orthogonal component of $\boldsymbol{F}$ on the coordinate axis x_1 [resp. x_2] [resp. x_3],
- $F = \mathrm{dist}(A, B)$,
- r_1, r_2, r_3 be the direction cosines of the straight line r individualized by A, B.

Obviously

$$F_1 = b_1 - a_1\,, \quad F_2 = b_2 - a_2\,, \quad F_3 = b_3 - a_3\,.$$

As a consequence, because of (1.2.4), it results

$$F_1 = F\,r_1\,, \quad F_2 = F\,r_2\,, \quad F_3 = F\,r_3\,. \tag{1.2.10}$$

REMARK 1.2.5 From (1.2.10) it follows that, if

- $\boldsymbol{F}$ and $\boldsymbol{G}$ are two oriented segments of $\mathfrak{R}^3$
- $\exists c \in \mathfrak{R}$ such that $G_1 = cF_1\,,\ G_2 = cF_2\,,\ G_3 = cF_3\,,$

then $\boldsymbol{F}$ and $\boldsymbol{G}$ are parallel. □

After that, let us consider any three non-aligned points $A = (a_1, a_2, a_3), B = (b_1, b_2, b_3), C = (c_1, c_2, c_3)$ of $\mathfrak{R}^3$, two by two distinct. We denote with α the plane individualized by points A, B, C. Any point $P = (p_1, p_2, p_3)$ of α can be obtained considering a point $Q = (q_1, q_2, q_3)$ of the straight line r_1 individualized by points A, B and subsequently reaching P on the line r_2 individualized by points Q, C. Because of (1.2.6) we have

$$q_1 = \frac{ma_1 + nb_1}{m+n}\ ,\ q_2 = \frac{ma_2 + nb_2}{m+n}\ ,\ q_3 = \frac{ma_3 + nb_3}{m+n} \tag{1.2.11}$$

where $(m, n) \neq (0,0)$. Still because of (1.2.6) we have

$$(1.2.12) \quad p_1 = \frac{hc_1 + kq_1}{h + k}, \quad p_2 = \frac{hc_2 + kq_2}{h + k}, \quad p_3 = \frac{hc_3 + kq_3}{h + k}$$

where $(h, k) \neq (0,0)$. Substituting the (1.2.11) in (1.2.12) we easily obtain

$$(1.2.13) \quad p_1 = \frac{ea_1 + fb_1 + gc_1}{e + f + g}, \quad p_2 = \frac{ea_2 + fb_2 + gc_2}{e + f + g}, q_3 = \frac{ea_3 + fb_3 + gc_3}{e + f + g}$$

where $(e, f, g) \neq (0,0,0)$. Evidently the algebraic linear system in the unknown real numbers *e, f, g, k*

$$\begin{aligned} k &= e + f + g \\ kx &= ea_1 + fb_1 + gc_1 \\ ky &= ea_2 + fb_2 + gc_2 \\ kz &= ea_3 + fb_3 + gc_3 \end{aligned}$$

must have non zero solution and then it must be

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ a_1 & b_1 & c_1 & -p_1 \\ a_2 & b_2 & c_2 & -p_2 \\ a_3 & b_3 & c_3 & -p_3 \end{vmatrix} = 0.$$

Such condition gives the equation of the plane α

$$(1.2.14) \qquad \alpha = \{(x_1, x_2, x_3) \in \mathfrak{R}^3 : ax_1 + bx_2 + cx_3 + d = 0 \text{ with } a, b, c, d \in \mathfrak{R} \text{ and } a^2 + b^2 + c^2 > 0\}.$$

Let us consider the plane α of equation $ax_1 + bx_2 + cx_3 + d = 0$, where $(a, b, c) \neq (0,0,0)$. If $Y = (y_1, y_2, y_3)$ belongs to α, it results $d = -ay_1 - by_2 - cy_3$ and then the equation of plane α can be written

$$a(x_1 - y_1) + b(x_2 - y_2) + c(x_3 - y_3) = 0.$$

Let now *a, b, c, d, a', b', c', d'* be any real numbers with $(a, b, c) \neq (0,0,0)$, $(a', b', c') \neq (0,0,0)$. We consider the plane α [resp. α'] of equation $ax_1 + bx_2 + cx_3 + d = 0$ [resp. $a'x_1 + b'x_2 + c'x_3 + d' = 0$]. We say that α and α' are *parallel* if $\alpha = \alpha'$ or α and α' don't have any point in common.

We have

[1.2.2] *Let α [resp. α'] the plane of equation $ax_1 + bx_2 + cx_3 + d = 0$ [resp. $a'x_1 + b'x_2 + c'x_3 + d' = 0$]. If a real number h exists such that a'=h a, b'=h b, c'=h c, then the planes α and α' are parallel and vice versa.*

Proof. To know if the planes α and α' have points in common, we must look for solutions of the system of algebraic linear equations

$$\begin{aligned} ax_1 + bx_2 + cx_3 + d &= 0 \\ a'x_1 + b'x_2 + c'x_3 + d' &= 0. \end{aligned}$$

To this aim we consider the matrix

$$C = \begin{bmatrix} a & b & c \\ a' & b' & c' \end{bmatrix},$$

which range obviously can't be zero. If the range of *C* is 2, obviously the planes have a straight line in common. If the range of *C* is 1, obviously the planes are parallel. The thesis follows. ◊

REMARK 1.2.6 Let us suppose that α and α' are parallel. Obviously if the complete matrix

$$\begin{bmatrix} a & b & c & d \\ a' & b' & c' & d' \end{bmatrix}$$

has range 1, then $\alpha = \alpha'$. If the complete matrix has range 2, then $\alpha \neq \alpha'$. □

Let

- r be an oriented straight line of $\mathfrak{R}^3$ having direction cosines r_1, r_2, r_3 ,
- *a, b, c, d* $\in \mathfrak{R}$ and such that $a^2 + b^2 + c^2 > 0$,
- α be the plane of $\mathfrak{R}^3$ having equation $ax_1 + bx_2 + cx_3 + d = 0$.

We say that *r* and α are *parallel* if $r \subseteq \alpha$ or *r* and α don't have any point in common.

We have

[1.2.3] *Let*

- *r be an oriented straight line of* $\mathfrak{R}^3$ *passing through the point* $A = (a_1, a_2, a_3)$ *and having direction cosines* r_1, r_2, r_3
- *a, b, c, d be real numbers such that* $a^2 + b^2 + c^2 > 0$,
- α *be the plane of* $\mathfrak{R}^3$ *having equation* $ax_1 + bx_2 + cx_3 + d = = 0$.

If

$$a\ r_1 + b\, r_2 + c\, r_3 = 0 \tag{1.2.15}$$

r and α *are parallel, and vice versa. Moreover if*

$$a\ a_1 + b\, a_2 + c\, a_3 + d = 0$$

then $r \subseteq \alpha$.

Proof. Since (1.2.6) a point $P = (p_1, p_2, p_3)$ belonging to *r* and to α must satisfy the conditions