

## *SUBJECT MATTER*

This chapter 1 opens the study of the *Mathematical Analysis*. First of all, they are clarified some basic principles of the mathematical logic. Still preliminarily, the significance of the main symbols used in the discipline (as  $\in, \notin, \subseteq, \subset, \cup, \cap, \exists, :, \Rightarrow, \Leftrightarrow, \emptyset$ ) is clarified.

The study begins by defining the *set* and the *operations* on sets. After that, there are given the definitions of *cover, partition, Cartesian product*.

After such basic definitions, there are defined the *relations* on sets. Particular attention is devoted to *equivalence relation* (and to *quotient set*) and to *partial* or *total order relation*. For ordered sets, there are given the definitions of *upper bound, lower bound, max, min, least-upper-bound property, inf*.

After that, it is defined the *function*, also called *single-valued relation* or *mapping* or *transformation* or *operation* or *correspondence* or *application*. Still, there are defined the *reversible function*, the *inverse function*, the *one to one* (or *biunique*) *correspondence*, the *restriction function*, the *extension function*, the *composite function*, the *characteristic function*. In particular, the one to one function allows to define the *equivalent sets*, the *finite set*, the *infinite set*, the *denumerable* (or *enumerable*) *set*, the *countable set*.

The basic study of sets is completed by giving some useful definitions and properties of the *algebra of sets*. Precisely, after the definition of *field*, there are detailed the *field axioms* and all the statements which they imply. Afterwards, there are given the definition of *ordered field* and all the statements which it implies.

The study of sets is completed by the construction of the most important set, that is the *real number set*  $\mathbb{R}$ . Such construction can be performed only using the set theory. However, the real number set is often constructed beginning by the *natural number set*  $\mathbb{N} = \{1, 2, 3, \dots\}$ . In such way, the definition and the properties of  $\mathbb{N}$  are assumed axiomatically, as primitive notions. After that, the natural number set is provided with the *laws of addition and multiplication*, and with the *order relation*.

Using  $\mathbb{N}$  as foundation, we can build a new set  $Q$ , called *rational number set*, which can be considered an extension of  $\mathbb{N}$ . Preliminary, it is appropriate to build the set  $Q^+$ , called *positive rational number set*. For this purpose, we provide  $\mathbb{N} \times \mathbb{N}$  with the equivalence relation

$$R = \left\{ \left( (n_1, n_2), (m_1, m_2) \right) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) : n_1 m_2 = n_2 m_1 \right\}.$$

In doing this, to indicate the generic ordered pair of natural numbers  $(n, m)$ , usually you prefer to use the  $\frac{n}{m}$  notation. The equivalence relation  $R$  divide  $\mathbb{N} \times \mathbb{N}$  into equivalence classes, in such a way that  $\frac{n_1}{n_2} \equiv \frac{m_1}{m_2}$  if and only if  $\frac{n_1}{n_2}$  and  $\frac{m_1}{m_2}$  belong to the same class. The family of all such equivalence classes is a family of nonvoid pairwise disjoint sets and its union is  $\mathbb{N} \times \mathbb{N}$  and then is a partition of  $\mathbb{N} \times \mathbb{N}$  (called the *quotient set* of  $\mathbb{N} \times \mathbb{N}$ ). Well,  $Q^+$  is defined as such *quotient set*. It is provided with the laws of addition and multiplication and with an order relation. From  $Q^+$  we can build a new set  $Q$  (containing  $Q^+$ ), which we call *rational number set*. Analogously, we can build a new set  $I$  (containing  $I^+$ ), which we call *integer number set*. Precisely, we create a new number 0 called *zero*, distinct from the positive rational numbers. We also create numbers which are distinct from the rational positive numbers as well as distinct from zero, and which we call *negative rational numbers*. The set consisting of all negative rational numbers, is denoted by  $Q^-$ . We call *rational number set*, and denote by  $Q$ , the set consisting of all positive rational numbers, of 0, and of all negative rational numbers. We call *integer number set*, and denote  $I$ , the set consisting of all positive integer numbers, of 0, and of all negative integer numbers. Obviously  $I \subset Q$ .  $Q$ , provided with the laws of addition and multiplication and with an useful order relation, is an *ordered field*.

The set  $Q$ , although it is an orderly field, has serious insufficiencies. The most insufficiency of  $Q$  is the lack of *completeness*. Precisely, we prove that  $Q$  has not the *least-upper-bound property*, also called the *completeness property*. The *completeness* is the fundamental property that makes possible the *Mathematical Analysis*. For this reason, it was created from  $Q$  a new number set, called *real number set*, and denoted  $\mathbb{R}$  (or  $] -\infty, +\infty [$ ). One of the ways to create  $\mathbb{R}$  is due to *Dedekind*. He called *cut*, or *real number*, any nonempty subset  $x$  of  $Q$  such that  $(p \in x, q \in Q, q < p) \Rightarrow (q \in \alpha)$  and  $(p \in x) \Rightarrow (\exists r \in \alpha \text{ such that } p < r)$ . About the order relation on  $\mathbb{R}$ , we say that  $x$  is *less than*  $y$  (or that  $y$  is *greater than*  $x$ ) and we write  $x < y$  (or  $y > x$ ) if  $x$  is a proper subset of  $y$ . After that, we prove that the ordered set  $\mathbb{R}$  has the *least-upper-bound property*. Moreover, providing  $\mathbb{R}$  with suitable operations of *addition*, *multiplication*, *subtraction*, *division*, we can demonstrate that  $\mathbb{R}$  is a *complete ordered field*. To complete the study of  $\mathbb{R}$ , we demonstrate its *Archimedean property* and define the *rational real numbers*, the *irrational real numbers*, the *absolute value* of a real number (and detail its properties).

It is convenient to adjoin to  $\mathbb{R}$  two new elements, *which are not real numbers*. These objects, called *infinity* and *minus infinity*, are respectively denoted  $+\infty$  (or simply  $\infty$ ) and  $-\infty$ . The set  $[-\infty, +\infty] = \mathbb{R} \cup \{-\infty, +\infty\}$  is called the *extended real number set*. We provide it with an order and with operations of addition and of

multiplication.

As useful tools, there are defined

- the *open interval*  $]a, b[$  having *left endpoint*  $a$  and *right endpoint*  $b$
- the *closed interval*  $[a, b]$  having *left endpoint*  $a$  and *right endpoint*  $b$
- the *integral power* of a real number (with the detail of its properties)
- the *n-th root* of a real number
- the *factorial* of  $n \in \mathbb{N} \cup \{0\}$
- the *binomial coefficient*  $\binom{n}{k}$ ,

there are proven

- the Newton's binomial formula
- the density of  $Q$  in  $\mathbb{R}$

and it is discussed the *representation* (in various *bases*) of the numbers.

Another important number set is the set consisting of all ordered pairs of real numbers, called *complex number Set*. This set, provided with suitable operations of addition and multiplication, is a *field*. However, it *cannot be ordered*. Then, it makes no sense to talk about inequalities between complex numbers.

Introducing the imaginary unit  $i$ , any *complex number*  $(a, b)$  can take the *algebraic form*  $a + ib$ , where  $a$  is called the *real part* and  $b$  is called the *imaginary part*. After that, for any complex number there are given the definitions of *complex conjugate* and of *absolute value* (or *modulus*).

There is also another form of complex numbers, very much employed in various sectors of *Engineering*. It is called *trigonometric form* of the complex number. Precisely, with each complex number  $z = a + ib$  we associate a point with rectangular coordinates  $a$  and  $b$  in the complex plane. The  $x$ -axis along which  $a$  is reckoned is called *real axis* (or the *axis of reals*) and the  $y$ -axis along which  $b$  is reckoned is called *imaginary axis* (or the *axis of imaginaries*). Denoting by  $\rho$  the length of the radius vector of the point  $z$  and by  $\vartheta$  (for  $z \neq 0$ ) the angle (expressed in radians) formed by the radius vector with the positive half-axis  $x$  we clearly have  $a = \rho \cos \vartheta$ ,  $b = \rho \sin \vartheta$ ,  $\rho = \sqrt{a^2 + b^2} \geq 0$ . Hence, since  $z = a + ib$ , we have  $z = \rho(\cos \vartheta + i \sin \vartheta) = [\rho, \vartheta]$ . The trigonometric form allows to perform in a very simple way the operations of multiplication, division, calculation of integral power, calculation of  $n$ th roots. In particular, there are given all the  $n$ th complex roots of the positive real unit.